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The purpose of this interim unit is to assist teachers and pupils in making the transition to a contemporary mathematics program from the present program. To enable students to make this transition more easily, each unit is based on familiar concepts, develops gradually, and is directed toward certain fundamental principles. Children should be given an opportunity to discover many of these new concepts. Teachers may guide discovery by asking thought-provoking questions and using concrete materials.

The three units should be covered in a minimum of six weeks and a maximum of eight weeks. Each unit is organized with a teacher objective and backgroud information. Teaching procedures are divided into the teacher's exploration of the unit followed by pupil exercises.

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FOR GRADE 4

Sets, Numbers and Operations

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AN INTERIM UNIT
IN CONTEMPORARY MATHEMATICS
FOR GRADE 4

Sets, Numbers, and Operations

Prepared
by

George R. Vojtko

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San Diego City Schools
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Unedited Edition, January 1965

TO THE TEACHER

The teaching of arithmetic is undergoing many changes due to the impact of contemporary mathematics programs which have been conducted nationally during the past several years. It is anticipated that this trend will continue for some time in the future.

Developing an arithmetic curriculum during this period of rapid change poses many problems: new terms are being introduced, techniques of teaching are being modified, and arithmetic texts are continually being revised and updated. Many of the contemporary mathematics concepts are not included in the San Diego City Schools basic arithmetic text; however, they will appear in the next state textbook adoption which will be made available in September, 1965.

It is neither possible nor desirable to write directions and suggestions which will fit every teaching situation. Therefore, teachers should treat the content of this interim unit merely as a suggested guide to be implemented by their own resourcefulness.

Adjust the teaching procedure and activities described in this publication to your pupils and to your proven teaching techniques. Although teaching contemporary mathematics may be new to some of you, teaching children is your specialty.

Wm. H. Stegeman

William H. Stegeman
Assistant Superintendent
Curriculum Services Division

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GLOSSARY. 134

PURPOSE

The purpose of this interim unit is to assist teachers and pupils in making the transition to a contemporary mathematics program and to the state textbooks that will be available in the fall of 1965. Experience has shown that any introductory work during the year preceding the introduction of a "new" program has made the first year considerably easier and more profitable for both teachers and pupils. Much of the content in this unit closely parallels the content of the textbook series,"Modern Arithmetic Through Discovery,"Silver Burdett Company.

Each concept presented in this unit is (a) based on familiar concepts, (b) developed gradually, and (c) directed toward certain fundamental principles. Pupils will not master these concepts immediately. However, experience and practice during this interim period will promote better understanding of the relationship emphasized in the new program.

SUGGESTIONS FOR USING THE UNIT

There are two ways in which this unit may be used:

1. To plan to complete the current state textbook (Winston) in the spring and teach the concepts in this unit in the remaining weeks of the year.

More Able Group. Upon completing the work in the present state textbook (Winston), begin teaching the concepts suggested in this unit. It is anticipated that many teachers of grade four will start during the month of March or earlier. The teaching time should take from 6 to 8 weeks. If this unit is completed before the close of the school year, the teacher is encouraged to begin teaching from the next grade-level state text, adapting materials to fit the ideas included in this unit.

Less Able Group. Whenever this group completes the work in the present state textbook, introduce them to this unit. Plan to devote 6 to 8 weeks to the unit. In some instances, it will be necessary to omit some of the content in the Winston with this group.

2. To relate the teaching of concepts in this unit with the current state textbook during the remainder of the year.

In either case, it will be the teacher's responsibility to (a) become acquainted with the entire unit before any teaching is attempted, (b) spend a minimum amount of time on Chapters I and II and devote the major effort to those concepts in Chapter III, and (c) paraphrase much of the TEACHING PROCEDURE in the exploration phase, making modifications where necessary.

ORGANIZATION OF THE UNIT

Objective. The objective for teaching each concept is stated and includes the goal in understanding which the teacher should expect to achieve.

Background Information for the Teacher. The information under this heading is briefly stated and serves as a source of information which will be helpful to the teacher.

Teaching Procedures. This category is divided into two phases:

1. Exploration. This is largely an oral phase of the unit in which the teacher is provided with (a) questions which may be asked to promote inquiry and discussion, (b) examples which may be used to illustrate an idea, and (c) suggestions for use of concepts or semiconcrete materials. This is done prior to the time pupils are assigned any written exercises and is especially important with the less able groups.

2. Pupil Exercises. This phase of the unit is devoted to written exercises which are primarily aimed at reinforcing those skills and principles learned in the oral phase and practice phase of the lesson. After a concept is presented and developed, each pupil is provided with a worksheet. The same worksheet, with answers, is included in this guide. With less able pupils, the written work may be done in connection with oral reading, discussion, and use of illustrative, concrete materials.

DISCOVERY APPROACH

Children should be given an opportunity to "discover" many of the principles* of arithmetic which they will be expected to learn and use. Ideas which pupils discover for themselves, individually or through class discussion, make sense to them; rules which are committed to memory by rote methods are soon forgotten.

The discovery approach in teaching arithmetic can be fostered by the teacher who:

(a) Asks thought-provoking questions.

(b) Provides opportunities for discovery of the principles and structure of arithmetic through exploration with concrete materials.

(c) Encourages the search for patterns in a variety of number and geometric situations.

(d) Encourages studious inquiry through discussion of various ways of thinking about mathematical situations.

---

*See GLOSSARY, page 134, for a brief discussion of the mathematical terms used in this unit.

(e) Provides opportunities for testing of hypotheses through application of a theory.

(f) Withholds formal verbalization of the basic principles until they are understood operationally and used intuitively, allowing pupils to state ideas informally in their own language.

(g) Assists the pupil in making generalizations as a result of investigation.

When pupils learn arithmetic through their own discoveries, they develop (a) habits of logical thinking, (b) insight into social-mathematical situations, (c) accuracy and efficiency, and (d) ability to reconstruct an operation in arithmetic without relying wholly upon memory.

Results of recent research indicate that learning is improved when the generalization of a concept is discovered by the pupil rather than "handed" to him before he has had an opportunity to understand and use the skill related to the concept. This means that formal statements of generalizations should not be introduced prematurely but withheld until the concept has been developed and understood at an intuitive level.

SUGGESTED WEEKLY TIME ALLOTMENTS

		Weeks	
	Chapter	Minimum	Maximum
I.	Sets (pages 4-15)	1	$1\frac{1}{2}$
II.	Numbers (pages 16-53)	$1\frac{1}{2}$	2
III.	Operations (pages 54-133)	$3\frac{1}{2}$	$4\frac{1}{2}$
	Total Number of Weeks	6	8

CHAPTER I

SETS

Many mathematical concepts were started thousands of years ago. Some of them probably were an outgrowth of man's need for a system which would help him keep account of such possessions as a flock of sheep. Others may have developed from his need to determine the size of objects and to find his position in relation to his surroundings.

When primitive man made marks or impressions on the ground and paired a mark on the ground with each of his animals, he developed a useful accounting system. This system helped him determine whether he had the same number of animals he started with, whether he had more animals, or whether he had fewer. Later, man developed a more efficient system of keeping an account of possessions; he designed a set of symbols which enabled him to name the number of them. Over the centuries other ideas that make communication and work easier were added to the "storehouse" of mathematical concepts. Mathematicians studied and refined the ideas, developing an abstract structure of interrelated concepts that is applicable in widely diverse situations.

Among the relatively new and very useful concepts in mathematics are those included in the study of SETS. The development of a formal theory of SETS began in the late years of the nineteenth century by Georg Cantor, a German mathematician. Concept of SETS were used to solve such difficult problems as: (1) "How many whole numbers are there?" (2) "How many points are there on a circle?" (3) "How many numbers are there between 1 and 2?" Solving equations, drawing graphs, describing geometric figures, etc., becomes simpler by using concepts and language of SETS.

Because the language of sets is an effective way to express ideas not only in mathematics but also in many other fields of knowledge, the study of sets is becoming increasingly important. The formal theory of sets is abstract and not started at the elementary school level. However, some of the simplest ideas and language are introduced to clarify concepts and facilitate oral and written communication.

In this unit, an elementary approach to the concept of sets is presented in terms of selected oral language and written symbolism. The word set is used to mean a "definite collection of things, objects, or ideas." When we speak of a set, we describe it precisely so that we can tell whether or not any given object or idea belongs to the specified set. The objects or ideas contained in a given set are called elements or members of that set.

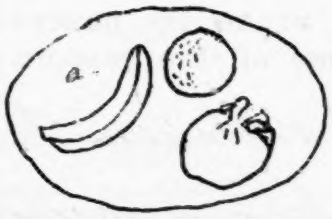
While there are many mathematical reasons for beginning the study of mathematics with the Concept of Sets, another powerful motivation for starting with sets is that "sets of objects" can be experienced with the senses and they can be easily manipulated. They afford the immediate concreteness that elementary pupils need.

A. Concept of Sets

Objective. To become better acquainted with the idea of a set and to begin building a "set language."

Background Information for the Teacher. To help us communicate our ideas about sets, we can describe sets in a number of ways:

1. Sometimes we draw pictures, using an outline to show that we are considering the set whose members are the objects pictured.



   Such pictures help us to communicate both the idea of a set and to formulate a description of its members. Picturing sets is used to a considerable extent in working with children.

2. We can describe sets in words, for example, the set whose members are an orange, an apple, and a banana; our class; and the set of odd numbers from one through nine.

3. Frequently, we list or tabulate the elements of a set within braces. For instance, the symbol {1, 2, 3, 4, 5} is simply a name for the set described by the phrase "the set of consecutive counting numbers from one through five." {1, 3, 5, 7, 9} is a listing or tabulation of the set of odd numbers from one through nine.

We can describe the set which contains no elements. Consider the set which contains all United States senators under sixteen years of age. Clearly, there are no members in this set which will match the description. The set which has no elements is called the empty set. Using the brace notation, the empty set is represented by the symbol { }.

It is sometimes convenient to use a capital letter as a temporary name for a set. The statement A = {1, 2, 3, 4, 5} is read "A is the set of consecutive numbers from one through five."

It is important that we be able to translate a description into a set tabulation, but we must also be able to change a tabulation into a description. For example, the tabulation {a, e, i, o, u} can be described as (1) "the set of vowels in the English alphabet" or (2) "the set whose members are a, e, i, o, and u."

TEACHING PROCEDURE

Exploration. The classroom is an ideal setting for opportunities to talk about sets. Ask the pupils to describe different sets which are clearly visible to all pupils. Some of the responses might be:

(1) The set of chairs in the front row in our room.

(2) The set of maps at the front of our room.

(3) The set of books on the top shelf of our bookcase.

Refer to an object that does or does not belong to each set named and ask if it is a member of the set. For example, "Does my desk belong to the set of chairs in the front row? How do you know?"

On the chalkboard, write the numerals 2, 4, 6, 8. Ask the pupils to describe this set. Some of the responses might be:

(1) The even numbers two, four, six, eight.

(2) The set of even numbers beginning with two and ending with eight.

(3) The set of even numbers between one and nine.

Next, discuss with your pupils that a convenient way of referring to a set is to use braces. In so doing, you will be tabulating or listing a set. For example, show the following tabulated sets on the chalkboard:

M = {1, 2, 3, 4, 5}

R = {a, b, c}

S = {Tuesday, Thursday}

*N = {   }

Now, ask the pupils to describe the sets M, R, S, and N shown above. Some of the responses might be:

M - The set of counting numbers less than six. The counting numbers beginning with one and ending with five.

R - The set whose members are the first three letters of the alphabet.

S - The set of days of the week beginning with the letter T, or the set whose members are Tuesday and Thursday.

N - The set consisting of "me."
The set whose member is "....."

To develop the idea of the empty set, write the symbol { } on the chalkboard. Ask the pupils to describe this set. (Note: the idea of a set having no members may serve as a basis for discussion. An empty

---

*Write your name within the braces.

wallet, an empty specimen jar, or perhaps an empty pocket may be used to develop insight into such a set.) Some of the responses might be:

(1) The set without any members.
(2) The set which is empty.
(3) The empty set.

Next, ask the pupils to give examples or to describe the empty set. Some examples are:

(1) The set of boys three years of age in grade 3.
(2) The set of girls nine feet tall.
(3) The set of even numbers between 10 and 12.

SUMMARY OF BASIC IDEAS

1. A set is a definite collection of things, objects, or ideas.
2. Each object or idea in a given set is a member or element of the set.
3. Pictures are sometimes used to illustrate a set.
4. Words are used (without braces) when describing a set.
5. Braces are used when tabulating a set.
6. A set may have a limited number of members, an unlimited number of members, or have no members; that is, it may be empty.

Pupil Exercises. Provide each pupil with worksheet No. 1.

Unless the group is advanced, read the exercises with the pupils and have them respond orally. Then have them do the exercises independently. Explain example 12 and use of the asterisk and footnote.

Worksheet No. 1

LIST THE MEMBERS OF EACH SET. BRACES ARE PROVIDED FOR YOU. TRACE OVER THEM FOR PRACTICE.

1. The first five letters of the English alphabet. {a, b, c, d, e}
2. The counting numbers that are greater than 10 and less than 15. {11, 12, 13, 14 }
3. The numbers counting by 2's, beginning with 2 and ending with 14. {2, 4, 6, 8, 10, 12, 14 }
4. The numbers counting by 5's, beginning with 10 and ending with 45. {10, 15, 20, 25, 30, 35, 40, 45}
5. The days of the week whose names begin with the letter M. {Monday }
6. The boys and girls in your class who are less than six years old. { }
7. The planets in the solar system whose names begin with the letter V. { Venus }
8. The months of the year whose names begin with the letter J. {January, June, July }
9. The numbers between 30 and 35. { 31, 32, 33, 34 }
10. The seasons of the year. {spring, summer, autumn, winter}
11. The twenty-ninth letter of the English alphabet. { }
12. The letters which are in the name of your school but are not in your last name.* Answers will vary. { }
13. The even numbers between 15 and 19. { 16, 18 }
14. The odd numbers between 7 and 8. { }

---

*Do not list any letter more than once.

B. One-to-One Pairing

Objectives. (1) To provide a useful method for determining the relationship between sets. (2) To determine if given sets are equal, not equal, or equivalent.

Background Information for the Teacher. The idea of pairing objects is not new to any teacher. Moreover, it is quite likely that many of your pupils also have had informal experiences with pairing objects. For example, when a pupil is asked to distribute a sheet of paper to each member of the class, he is aware of a relationship between the paper and the pupil. When he matches a sheet of paper with each pupil in the class he has established a one-to-one correspondence; each pupil in the classroom has a sheet of paper and for each sheet of paper that was distributed there is a pupil.

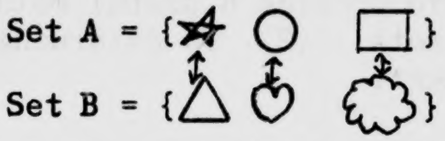
When a one-to-one correspondence exists between two sets, the sets are said to be equivalent. Centuries ago when shepherds kept account of their sheep by pairing a pebble with each member of a flock they were using the concept of equivalent and non-equivalent sets. If the sets were matching sets, that is, if there was a one-to-one correspondence between pebbles and sheep, they knew that all the sheep had returned from pasture. When there were pebbles "left over" they knew that some sheep were missing; fewer sheep were in the flock than originally. When there were sheep "left over" after the pairing process was completed, they knew there were more sheep in the flock than at a prior time.

A comparison of any two sets can be made by using the pairing process. Thus, one can determine, without counting, which of the sets has more members, which has fewer members, or whether the sets are equivalent.

Note that when a one-to-one correspondence exists between two different sets, the sets are equivalent, not equal. Equal sets are sets that have exactly the same members. For example, suppose we designate the set of children who serve on the clean-up committee as Set C. Lynn, Bob, and Sara are the members of the clean-up committee. We can say that the set of children on the clean-up committee equals the set of children consisting of Lynn, Bob, and Sara. Or, since we named this set C, we can write: C = {Sara, Bob, and Lynn}. We are using different names to specify the identical set. The order of listing the members is irrelevant.

It is so obvious that equal sets are also equivalent that discussion of the idea sometimes seems confusing. The following illustrations may serve to clarify it.

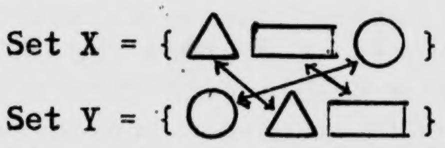
1. Can Set A be placed in one-to-one correspondence with Set B?
   Is Set A equal to Set B?



The members of Set A and Set B can be paired and we "come out even." Since the sets are in one-to-one correspondence, they are equivalent.

The members of Set A are a star, a circle, and a square. The members of Set B are a triangle, a heart, and a flower. The sets are not the same, therefore they are not equal.

2. Can Set X be placed in one-to-one correspondence with Set Y?
   Is Set X equal to Set Y?



The members of Set X and Set Y can be paired and we "come out even." Since the sets are in one-to-one correspondence, they are equivalent.

The members of Set X are a triangle, a rectangle and a circle. We find pictures of the same triangle, rectangle, and circle in Set Y. The sets are identical, therefore they are equal.

The distinction between equivalence and equality of sets has been pursued at some length because it is basic to the concept of equality that is developed throughout the mathematics program. The equal sign (=) in mathematics is the mathematical symbol for "is identical with." When we consider the number property of the equivalent sets in the examples above, we see that the number of elements in each is, in fact, identical.

We could use the word three, the Arabic numeral 3, the Roman numeral III, and other names for the number of each set, and correctly say that three = 3, etc., because we are naming the same number. Since the concepts of equality and inequality are fundamental in mathematics, it is important for the teacher to use the words equivalent and equal correctly in reference to sets of objects.

TEACHING PROCEDURE

Exploration. (One-to-One Pairing) Ask your pupils to pretend that they know nothing about counting and that they are without a written language. Solve the following problems: *

1. "Show us how you can tell, without counting, whether or not the set of fingers on one hand is equivalent to the set of fingers on the other hand."

---

*Answers will vary.

They are not the same fingers, but they can be matched one-to-one.

2. When we all leave the room for relief or recess, how can we tell that upon returning, all pupils can be accounted for?"

Before leaving, check to see what chairs are vacant. Upon returning, if the same chairs are vacant, we can assume that all have returned.

(Discuss the fact that we are assuming only that no one has entered the room while we were out.)

3. Suppose you are interested in determining whether there are more girls than boys in your classroom. What would you do?"

Ask the girls to line up on one side of the classroom and the boys to line up on the opposite side. Then match or pair each girl with a boy. If the sets match one-to-one, that is, are equivalent sets, the number will be the same; if they do not match, there may be more boys than girls or more girls than boys.

4. Provide each pupil with a token (strip of paper, bottle cap, pebble, marble, or other small object. When the pupils leave the room for the playground one by one, have each pupil deposit a token into a container which has been placed by the door. Upon returning, have each pupil pick up one token. With the activity carried out (providing all pupils have returned), ask the following: "How does this compare with the way primitive man kept account of his sheep? With the distribution and collection of report cards?"

The method is the same; the examples are similar.

Exploration. (Equal sets). On the chalkboard write a verbal description such as "the set of national holidays during the month of February." Ask the pupils, "Name the members of the set." The responses might be:

{ Lincoln's Birthday, Washington's Birthday }

Next, point out that although it is possible to use words, pictures, or braces to describe sets (using commas between members to eliminate confusion between two or more elements), capital letters are used to name sets. Refer to the set {Lincoln's Birthday, Washington's Birthday}; label the set as A. The set should now be represented as:

A = {Lincoln's Birthday, Washington's Birthday}

Make it clear that the choice of letter name is an arbitrary matter in that A could just as well have been Q. Explain that a verbal description of A would be: "A is the set whose members are Lincoln's Birthday and Washington's Birthday," or "A is another name for the set whose members are Lincoln's Birthday and Washington's Birthday."

On the chalkboard show that a set could be tabulated (listed) in other ways. For example:

(1) A = {Lincoln's Birthday, Washington's Birthday}

(2) A = {Washington's Birthday, Lincoln's Birthday}

Lead the children to conclude that each of these tabulations names the same set. Ask the pupils to describe A, example (1) and example (2) above. Responses might be:

(1) A is the set whose members are Lincoln's Birthday and Washington's Birthday.

(2) A is the set whose members are Washington's Birthday and Lincoln's Birthday.

Next, ask, "Do examples (1) and (2) describe the same set? Yes What can we say about the order of listing the members of each of the sets?"

The order of listing the members in the set does not change the description.

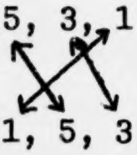
To develop the idea of "equality," write the following example on the chalkboard:

M = {5, 1, 3}

N = {1, 3, 5}

Explore with your pupils using (a) the recent discovery that the order of listing the members does not change the set described and (b) the concept of one-to-one correspondence. Use the example already on the chalkboard; draw the lines, matching names for the same number.

M = {5, 3, 1}



N = {1, 5, 3}

Indicate to your pupils that when we write M = N, we mean that M and N are different names for the very same set.

Have the pupils consider:

A = {1, 2, 3, 4, 5}

B = {5, 4, 3, 2, 1}

Ask, "Why is it correct to say that A = B?" Some of the responses might be:

(1) A has the same members as B.

(2) The set described by A is the same set as that described by B.

(3) Each member of A can be matched with the same number in B.

Exploration. (Equivalent Sets). The notion of a pair of sets being in one-to-one correspondence leads to the idea of equivalence. Two sets which can be placed in one-to-one correspondence with each other are said to be equivalent. For example:

S = {□ △ ✩}

T = {⊠ ◇ ▷}

Ask, "What general statements can we say about Set S and Set T?" (Ask the pupils to keep in mind such ideas as (1) description of a set, (2) one-to-one correspondence (matching), and (3) equal sets.)

(1) S is the set whose members represent a square, a triangle, and a star.

(2) The description of S does not match the description of T.

(3) S and T can be matched one-to-one.

(4) S and T are not the same in all respects.

(5) Sets S and T are not equal.

Indicate to the pupils that when two sets can be matched one-to-one, they are said to be equivalent, whether or not the members are the same or different.

To develop the idea of equivalent sets using other examples, write the following on the chalkboard:

X = {a, b, c}

Y = {1, 2, 3}

Z = {□ ✩ ○ △}

Ask, "In comparing sets X, Y, Z (only two sets at a time may be compared) what can we say about them?" Some of the responses might be:

(1) X is equivalent to Y; Y is equivalent to X.

(2) Y is not equivalent to Z; X is not equivalent to Z.

(3) X is not equal to Y; Y is not equal to Z.

(4) X is not equal to Z; Z is not equal to X.

SUMMARY OF BASIC IDEAS

1. A comparison of any two sets may be made by using the process of pairing one-to-one.
2. Sets are in one-to-one correspondence when the members of one set can be matched, one-to-one, with the members of another set.
3. The word "equal," when referring to sets, is intended to mean "the same set as."
4. Sets may be assigned a capital letter name. For example, when we write A = B, we mean that _A_ and _B_ are different names for the very same set.
5. The order of the names or members of a given set is irrelevant when comparing two sets.
6. Sets are equal whenever each member of one set is also a member of the other set, in one-to-one correspondence irrespective of order, and there are no other members involved.
7. Sets are equivalent whenever each member of one set can be matched with the members of another set, the members being same or different.
8. If two sets are equal, they are also equivalent. However, if two sets are equivalent, they are not necessarily equal.

Pupil Exercises. Provide each pupil with worksheet No. 2.

Discuss the worksheet with the pupils before requiring independent work. Explain example 12 and the use of the asterisk.

USE THESE SETS TO ANSWER THE FOLLOWING QUESTIONS. WRITE "Yes" OR "No" IN THE BLANK SPACES.

A = { 4, 5, 7} B = {5, 4, 7} C = {7, 4, 5}

1. Does A = B? Yes
2. Does C = A? Yes
3. Does C = B? Yes
4. Can A, B, and C be matched one-to-one? Yes

USE THESE SETS TO ANSWER THE FOLLOWING QUESTIONS. WRITE "True," "False," OR "None" IN THE BLANK SPACES.

X = {a, b, c, d} Y = {c, b, k, a} Z = {k, c, t}

5. Z = X False
6. X = Y False
7. Y is equivalent to Z. False
8. Z is equivalent to X. False
9. Which sets are both equivalent and equal? None
10. X and Y can be matched one-to-one. True
11. Z is equivalent to Y. False
12. Which sets are equal? None

COMPLETE EACH QUESTION BY LISTING THE SET.

*13. Set X is equal to {t, s, r, d}. Think of a set that has the same number of members as X but is not equal to X. Call it Set Z.

Z = { }

14. Set M is equal to {12, 9, 8, 7, 3, 1}. Think of a set that is equal to M. Call it Set N.

N = { 12, 9, 8, 7, 3, 1 }

---

* Answers will vary.

CHAPTER II

NUMBERS

A. Concept of Number

Objective. To emphasize that a number is an idea which is associated with sets of objects.

Background Information for the Teacher. If we stop to think about the process of counting, we quickly realize that it is nothing more than establishing a one-to-one correspondence between an ordered set of names of the counting numbers and the set we are counting. (Counting numbers are: 1, 2, 3, 4, etc., that is, they begin with 1 and continue indefinitely. They are sometimes called natural numbers.) The last number named in the set of counting numbers gives us the number of the set we are counting. This process is an improvement over the simple one-to-one matching of sets. It tells us not only that the sets have the same number but also what the number is.

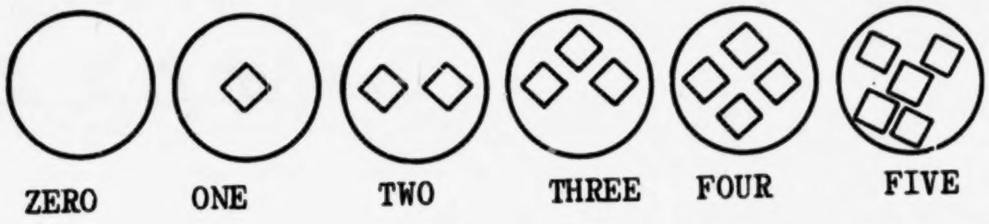
Suppose a pupil in your classroom is counting a set of objects such as a 12" ruler, compass, protractor, and a flannel board. To determine the "number" property or "how many" objects he has, he may begin by mentally matching the objects with the set of counting numbers. Thus,

Objects:	ruler	compass	protractor	flannel board
	↕	↕	↕	↕
Names of Counting Numbers:	1	2	3	4

After completing this matching, he can conclude that "there are 4 objects in this set."

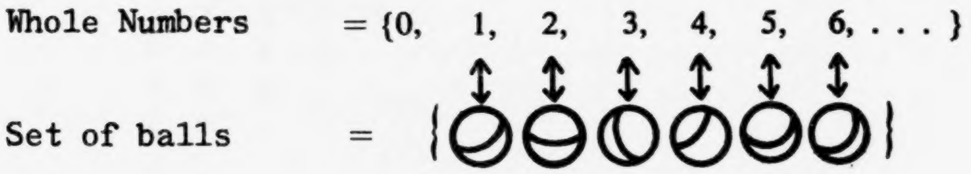
TEACHING PROCEDURE

Exploration. On the chalkboard represent a convenient model together with the number name such as the following illustration:

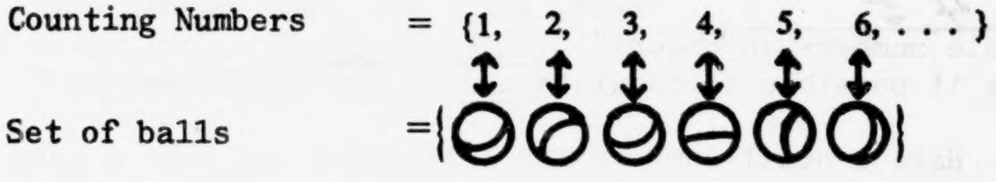


The pattern which develops shows clearly that the orderly sequence begins with the empty set and increases by "one more" member in each succeeding set. Clearly, this process can be continued indefinitely. Once an orderly system has been created and names and numbers assigned to these sets, we have the machinery with which to count.

To clarify the process of determining the number of members in a given set, inform the pupils that a standard set has to be established. (Whole numbers are sometimes called cardinal numbers.) The whole numbers are used in the following example:



Have the pupils notice that whenever we are counting a set containing one or more members, the first member is matched with the whole number one, the next with number two, etc.; an empty set would be associated with the whole number zero. Thus, the whole number (shortened to "number") of the set of balls is represented by the symbol 6. Illustrated in terms of one-to-one correspondence with the counting numbers we have the following:



Thus, the last number in the set of counting numbers matched with the set of balls indentifies the number of the set as represented by the symbol 6. Therefore, the standard set which determines the number of members in a set is the set of whole numbers. In counting "one and one more," the set of counting numbers is used.

As a group activity, to further develop the number idea ("how many" members are in a set), have the class compare the number of the set of red stripes in the United States flag with the number of the set of white stripes. 7 red and 6 white

Another activity might be to cut out fifty-five (55) felt objects of one uniform size and place them on the flannel board. (Teachers may prefer to use suitable representative markings on the chalkboard such as circles ○ , squares □ , or triangles △). Have several pupils separate the fifty-five objects into ten sets whose numbers are 8, 7, 5, 10, 6, 9, 4, 1, 3, and 2. Next have the pupils arrange the sets in order, so that each set contains one more object or member than the set before it. Thus, the orderly arrangement should look like this:

0	1	2	3	4	5	6	7	8	9	10
	x	x	x	x	x	x	x	x	x	x
		x	x	x	x	x	x	x	x	x
			x	x	x	x	x	x	x	x
				x	x	x	x	x	x	x
					x	x	x	x	x	x
						x	x	x	x	x
							x	x	x	x
								x	x	x
									x	x
										x

Discuss where the empty set should be placed in an orderly arrangement of sets. Help pupils to see that the empty set is logically a part of the arrangement. It precedes the set with one member since the set with the number one has one more member than the empty set.

Point out to the class that so far they have been working with sets of objects and the set of whole numbers. Now they will be working with sets of numbers and not the objects. Ask, "Who would come to the chalkboard and tabulate the whole numbers between 3 and 6?" { 4, 5 } Next, ask, "Who could tabulate the first three whole numbers in order?" { 0, 1, 2 } Now raise the question, "Is it possible to tabulate all of the whole numbers?"

Have a pupil come to the chalkboard and make a long list of the whole numbers beginning with 0. Lead the class to discover that no matter where the list stops, there will always be another number greater than the last one listed or tabulated. Explain that a list of all the numbers would be endless. Now refer back to the question that was raised, "Is it possible to tabulate all the whole numbers?"

No, because there are more numbers than can be counted.

Show the class a way to indicate that the list of whole numbers goes on endlessly; that is, {0, 1, 2, 3, 4, ...}. Explain that the three dots placed at the end of a number pattern indicate that the numbers go on without end. (With third- and fourth-grade pupils, avoid using the words "infinity" and "infinite.") Emphasize that three dots may indicate missing members if they appear in the midst of a pattern as shown in this example: 2, 4, 6, ..., 12, 14, 16.

Next, ask someone to count aloud the number of boys in the room. Ask, "Which of the whole numbers is not used in counting?" Zero Explain that the set of whole numbers beginning with one has a special name. It is called the set of counting numbers, that is, {1, 2, 3, 4, ...}.

SUMMARY OF BASIC IDEAS

1. "Number" is a property of every set; that is, a set may have no members, one member, two members, etc. Every set has a whole number associated with it.
2. The "number property" of a set is determined by matching the members of a set with a standard set.
3. The set of whole numbers is {0, 1, 2, 3, 4, ...}; that is, it begins with 0 and continues one more number in succession, indefinitely and consecutively.
4. The set of counting numbers is {1, 2, 3, 4, ...}; that is, it begins with 1 and continues indefinitely and consecutively.
5. A set may have a definite (finite) number of members or an indefinite (infinite) number of members.

Pupil Exercises. Provide each pupil with worksheet No. 3.

Before assigning this worksheet, make sure the pupils understand the instructions related to these exercises. Braces are provided.

Worksheet No. 3

IN COLUMN A, THE MEMBERS OF THE SETS ARE LISTED. WRITE THE NUMERAL THAT TELLS "how many" MEMBERS ARE IN EACH SET. WRITE YOUR ANSWER IN THE BLANK SPACE. THE FIRST ONE IS DONE FOR YOU.

	Column A	NUMBER OF MEMBERS
1.	M = { □, ✡, △, ▭ }	M = 4
2.	N = {a, b, c, d, e, f, g}	N = 7
3.	K = { }	K = 0
4.	R = {1, 2, 3, 4, ..., 10}	R = 10
5.	J = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}	J = 10
6.	X = {1, 2, 3, 4, ..., 39}	X = 39
7.	Z = {0, 1, 2, 3, ..., 29}	Z = 30
8.	W = {1, 2, 3, 4, ..., 99}	W = 99
9.	P = {2, 4, 6, 8, ..., 30}	P = 15
10.	Q = {3, 6, 9, 12, ..., 33}	Q = 11
11.	T = {1, 2, 3, 4, ..., 999}	T = 999
12.	S = {0, 1, 2, 3, 4, ..., 9999}	S = 10,000

LIST THE MEMBERS OF EACH OF THE FOLLOWING:

13. The set of whole numbers greater than 3 can be listed as {4, 5, 6, ...}. Three dots are used to indicate and so on. Now, use this method to list the members of the set of whole numbers greater than 45.

{ 46, 47, 48, ... }

14. The set of whole numbers less than zero. { }

15. The set of whole numbers between 107 and 109. { 108 }

16. The set of whole numbers between 64 and 65. { }

17. What is the first counting number? 1

18. The set of counting numbers less than 4. { 1, 2, 3 }

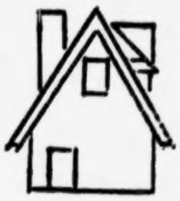
19. The set of counting numbers less than 8 and greater than 6. { 7 }

20. The last four letters of the English alphabet. { w, x, y, z }

B. Number-Numeral Distinction

Objective. To develop an understanding of the use of the term "number" which represents the numerical concept of quantity, and the term "numeral" which represents the symbol used to name the number.

Background Information for the Teacher. Children learn number names in connection with the number of objects in a set long before they can read or write the standard symbol or numeral. After they learn to write the numerals, they must realize that these symbols are names for numbers; they are not the numbers themselves. One way of expressing this is to say that numerals name or represent numbers. We interpret this in the same sense that each of the two visible symbols,



and house

represent a house. Most certainly, neither symbol is a "real" house.

> Number is an idea, an abstraction which arises from situations involving sets of objects.
>
> Numeral is a symbol by which we refer to a number.
>
> There are many ways of naming the same number, that is, there are many numerals for each number.

Perhaps the simplest way to explain the idea is to say, "Numerals are names for numbers." Write the word "Bob" on the chalkboard and ask the pupils what you have written. If some of them say "a boy," invite them to come to the chalkboard and play with "Bob." They will easily recognize that "Bob" is not a boy, but merely the name for a boy. Similarly, 2 is not a number but merely a visible name or symbol for a number.

If man lived alone in this world, the number idea would have been sufficient. But since man is a social being, he wishes to communicate his thoughts to other human beings. This is why numerals were invented.

Another point of interest which requires our attention and consideration is that a number idea or number may be expressed in many ways since a number has many names. For example, "Six," "VI," "4 + 2," "12 ÷ 2," and "3 × 2" are all names for the same number, since they are all numerals that name (or represent) the same number.

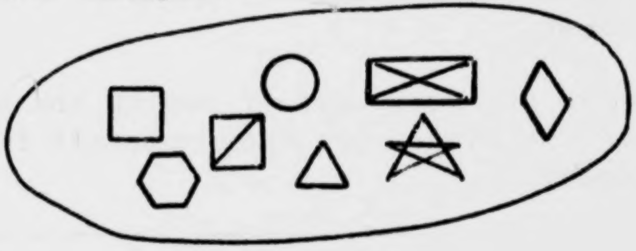
There may be some misgiving concerning the numerals "4 + 2" and "3 × 2." Yet,what number do you think of when you see "4 + 2" and "3 + 2"? In either case you think of the number we call "six." Hence, each of these communicates the idea of "six" and is therefore a numeral (or name) for the same number. The expression "4 + 2" expresses 6 as a sum and "3 × 2" expresses 6 as a product.

The study of arithmetic has too often meant merely the development of computational skills. Little consideration has been given to the nature of the "things" with which we compute. As an outgrowth of this emphasis, many feel an irresistable urge to "do an exercise" when they see an expression such as 4 + 2. This "urge to compute" or simplify compels some pupils to replace 4 + 2 by 6. However, 4 + 2 is a correct name for a number as is 6.

It is important for pupils to be able to write other names for 6. Sometimes the expanded names 4 + 2, 7 - 1, 24 ÷ 4, etc. are more useful than the concise form; pupils need to be able to find a useful expanded form as well as the most concise form.

TEACHING PROCEDURE

Exploration. On the chalkboard, draw a picture to represent a set of eight objects, for example:

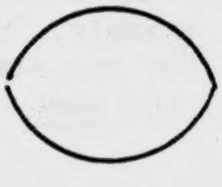


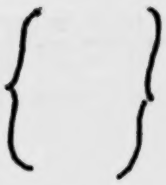
Ask pupils to come to the chalkboard and write different numerals to represent the number of the set. Some of the responses might be:

8, 7 + 1, 4 + 4, 4 × 2, 6 + 2

Lead the class to generalize that many numerals may be used to name the same number. Point out that the numeral 8 is the simplest numeral for the set illustrated above.

Continue the activity by asking for various numerals to name other sets pictured on the chalkboard. Include practice in writing numerals for the number of the empty set as shown below:





Some of the responses might be:

4 × 0; 20 - 20; 0 - 0; 8 - 8; 0; etc.

Inform the pupils that all of the written work at the board required symbols or numerals to represent the number idea.

Next, tell the pupils that you are thinking of the number "12." Ask two pupils to come to the chalkboard and have each pupil write several different numerals for the number "12." Responses for pupil A and pupil B might be:

Pupil A	Pupil B
6 + 6	7 + 5
8 + 4	9 + 3
10 + 2	11 + 1
13 - 1	17 - 5

Now, write an equal sign (=) between each of the two corresponding numerals. For example:

6 + 6 = 7 + 5
8 + 4 = 9 + 3
10 + 2 = 11 + 1
13 - 1 = 17 - 5

Help pupils realize that the equal sign indicates that two names are being used for the same number. Next, have several pairs of pupils name numbers in two different ways and use the equal sign to show that the two numerals they have chosen represent the same member. Encourage the use of a variety of numerals and operations. For example:

Number Idea	Numeral Used to Express the Number		
8	16 - 8	or	4 + 4
14	5 + 9	or	4 + 10
10	15 - 5	or	16 - 6
6	12 ÷ 2	or	2 × 3

Point out to the class that the idea of number may be communicated by spoken words as well as by written symbols. Thus, the concept that a number is an idea and that a numeral is used to communicate the idea will be reinforced.

SUMMARY OF BASIC IDEAS

1. "Number" is an "idea"--an abstraction which arises from a situation involving sets of objects.
2. "Numeral" is a symbol which is used to communicate the number idea; numerals name numbers.
3. A number has many names which can be written in many ways.
4. An equal sign (=) is used to show that two different numerals name the same number.

Pupil Exercises. Provide each pupil with worksheet No. 4.

Have the pupils read the instructions carefully before attempting the exercises independently. With the less able group, it may be necessary to interpret the instructions by doing several of the examples on the chalkboard.

Worksheet No. 4

"An equal sign is used between two numerals to show that they name the same number.

Examples: 8 + 5 = 13 27 = 13 + 14

The numeral 8 + 5 and the numeral 13 both name the number thirteen. Likewise, the numeral 27 and the numeral 13 + 14 both name the number twenty-seven."

FOR EACH NUMERAL IN COLUMN A, CHOOSE A NUMERAL FROM COLUMN B. SHOW THAT THE CHOSEN NUMERALS NAME THE SAME NUMBER.

	Column A	Column B	Names for the same number
1.	79	5 + 3	79 = 56 + 23
2.	16 - 16	12 + 5	16 - 16 = 37 - 37
3.	5 + 35	11 - 6	5 + 35 = 23 + 17
4.	12 - 7	17 - 8	12 - 7 = 11 - 6
5.	8	23 + 6	8 = 5 + 3
6.	29	23 + 17	29 = 23 + 6
7.	18 - 9	37 - 37	18 - 9 = 17 - 8
8.	13 + 4	56 + 23	13 + 4 = 12 + 5

IN THE BLANK SPACES, WRITE THREE DIFFERENT NAMES FOR THE NUMER NAMED AT THE LEFT. THE FIRST ONE IS DONE FOR YOU.*

9.	35	30 + 5	50 - 15	35 + 0
10.	73	70 + 3	50 + 23	80 - 7
11.	58	50 + 8	60 - 2	116 ÷ 2
12.	108	100 + 8	200 - 92	216 ÷ 2
13.	155	100 + 55	200 - 45	50 + 50 + 55
14.	99	90 + 9	100 - 1	99 + 0
15.	350	300 + 50	400 - 50	425 - 75
16.	987	900 + 80 + 7	900 + 87	910 + 77
17.	200	100 + 100	200 + 0	100 X 2
18.	504	500 + 4	510 - 6	0 + 504

---

* Answers will vary.

C. Order Relations

Objective. To show that our system of expressing number ideas is orderly and is useful in comparing numbers.

Background Information for the Teacher. Suppose a pupil opens a box of plastic or felt cut-out numerals and scatters them on the table. If he knew nothing about our numeration system, he would probably put them in any order, that is, 3, 4, 5, 1, 2, 6, etc. However, having some experience with our numeration system, he probably would arrange them in the order 1, 2, 3, 4, 5, etc.

The concept of order is very important in mathematics just as it is in any field, be it scientific or non-scientific. For example, given a set, we can often arrange the elements in some "agreed upon" order.

1. Natural order of the counting numbers:

1,	2,	3,	4, ... ,	n
↕	↕	↕	↕	↕
first	second	third	fourth	..., n^{th}

2. Order of the letters of the alphabet:

a,	b,	c,	d, ... ,	z
↕	↕	↕	↕	↕
first	second	third	fourth ...	twenty-sixth

The set of whole numbers has a natural order. They can be so arranged that each number is one greater than the preceding number: 0, 1, 2, 3, 4, 5, ...

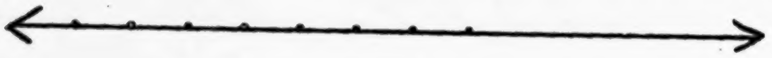
TEACHING PROCEDURE

Exploration. Indicate to the pupils that one way to show the orderly arrangement of the set of whole numbers is the "number line."

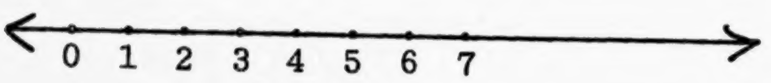
Simply represent a line as illustrated (a) with arrows on both ends indicating that it can go on and on as far as we want to extend it.

(a)

(b)

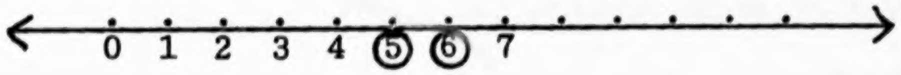


(c)

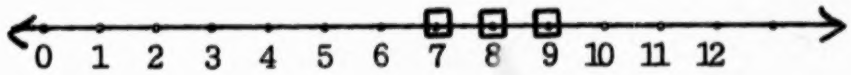


Then we mark on it, as in illustration (b), preceding page, a number of equally spaced marks or points. Starting at the left, we label the points with the whole numbers in order, that is, 0, 1, 2, 3, 4, 5, 6, 7, ... The arrows on the number line indicate we can extend it if we want to; the three dots ... at the end of our list of numbers indicate that we can extend it, also. Now the order of the whole numbers shows up clearly by the position of the labeled points. The point labeled 7 is to the right of the point labeled 5; 12 would be to the right of 11; 23 would be to the left of 27; etc. Thus we can see that as we move to the right, the numerical value increases; as we move to the left from a given point, the numerical value decreases. Because of this orderly arrangement, we can show that 27 is greater than 26; 29 is greater than 28; 35 is less than 36; 43 is less than 45; etc.

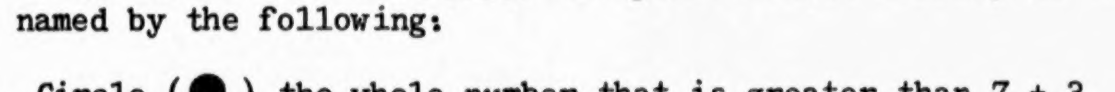
Now, ask a pupil to come to the chalkboard and draw circles around the numeral(s) that represent the whole number(s) less than 7 and greater than 4.



Next, have another pupil draw a box or square-shaped frame around the points that correspond to the whole numbers which are greater than 6 and less than 10 (extend the number line for this if necessary).



Construct another number line, this time beginning with twelve; have pupils write numerals to name the points on this line. Have someone circle the points that represent the whole numbers greater than 13 and less than 20, etc.

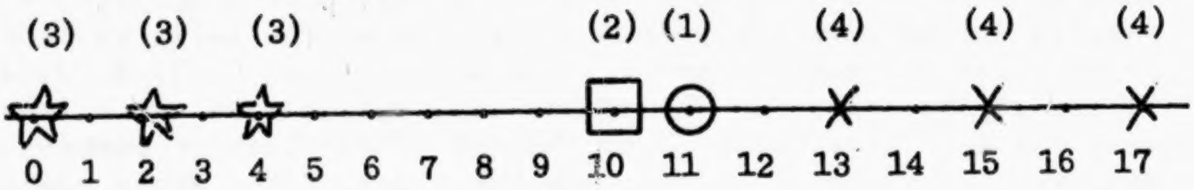


Vary this activity until pupils can readily work with situations similar to those mentioned above.

Now, construct a number line, showing equal unit segments, and show the whole numbers corresponding to the points on the line. Ask pupils to come to the chalkboard and locate the points which correspond to the numbers named by the following:

(1) Circle (●) the whole number that is greater than 7 + 3.
(2) Frame (■) the whole number that is 2 less than 14 - 2.
(3) Star (★) the even whole numbers that are less than 10 - 4.
    (0 is considered an even whole number.)
(4) (X) the odd whole numbers that are greater than 4 + 7.

Problems:



Proceed in this way to give pupils several opportunities to find other points on a number line.

SUMMARY OF BASIC IDEAS

1. The concept of order is important in mathematics just as it is in any field, be it scientific or non-scientific.
2. The set of whole numbers has a natural order.
3. The number line is one of many ways in which we can illustrate the orderly arrangement of whole numbers. It is useful in analyzing and solving many number situations.

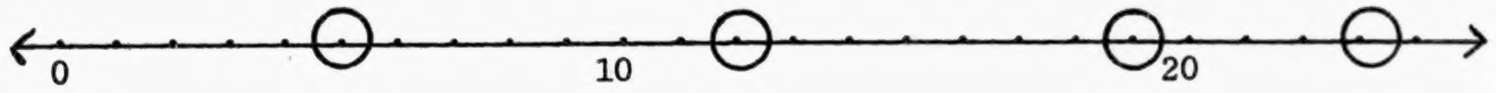
Pupil Exercises. Provide each pupil with worksheet No. 5.

Discuss the instructions with the pupils before assigning this worksheet. Use chalkboard illustrations, especially with the less able group, to clarify the instructions.

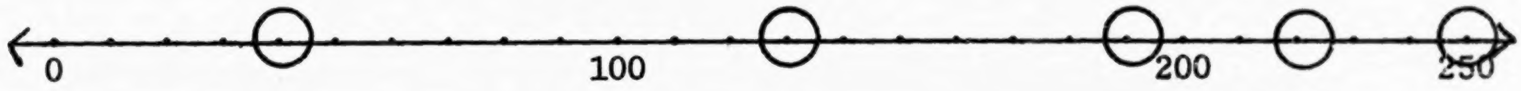
"Locating points on the number line"

DRAW A CIRCLE AROUND THE POINTS WHICH CORRESPOND TO THE SET OF NUMBERS NAMED IN EACH PROBLEM.

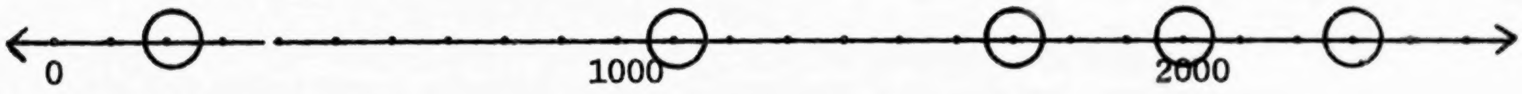
1. {5, 12, 19, 23}



2. {40, 130, 190, 220, 250}

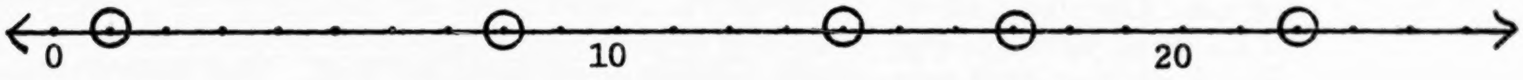

0
100
200
250

3. {200, 1100, 1700, 2000, 2300}



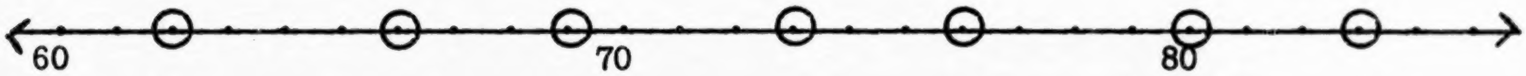
WRITE THE CORRECT NUMERAL BELOW EACH CORRESPONDING POINT WHICH IS CIRCLED.

4.



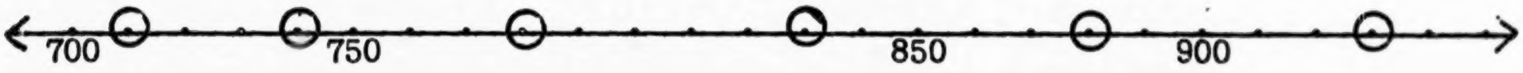
1 8 14 17 22

5.



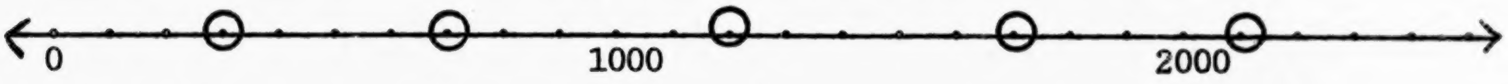
62 66 69 73 76 80 83

6.


700
750
850
900

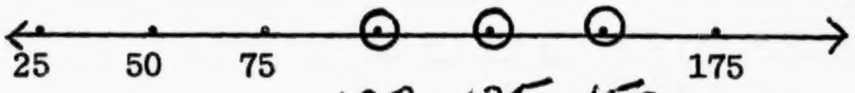
710 740 780 830 880 930

7.



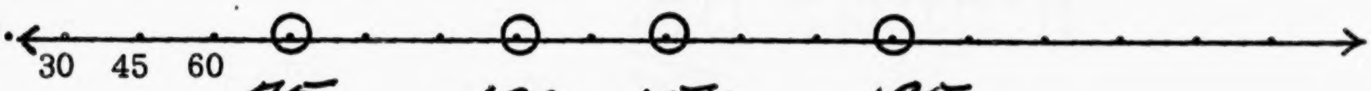
300 700 1200 1700 2100

8.



100 125 150

9.



75 120 150 195

D. Place Value (Decimal System)

Objective. To show that our system of numeration, because of the inherent place-value property, makes it possible for us to name any whole number without using more than ten distinct symbols or digits.

Background Information for the Teacher. The place-value scheme that our system utilizes has a base of ten. Base ten means that we group by tens. That is, given a set of objects (see illustration A) below:

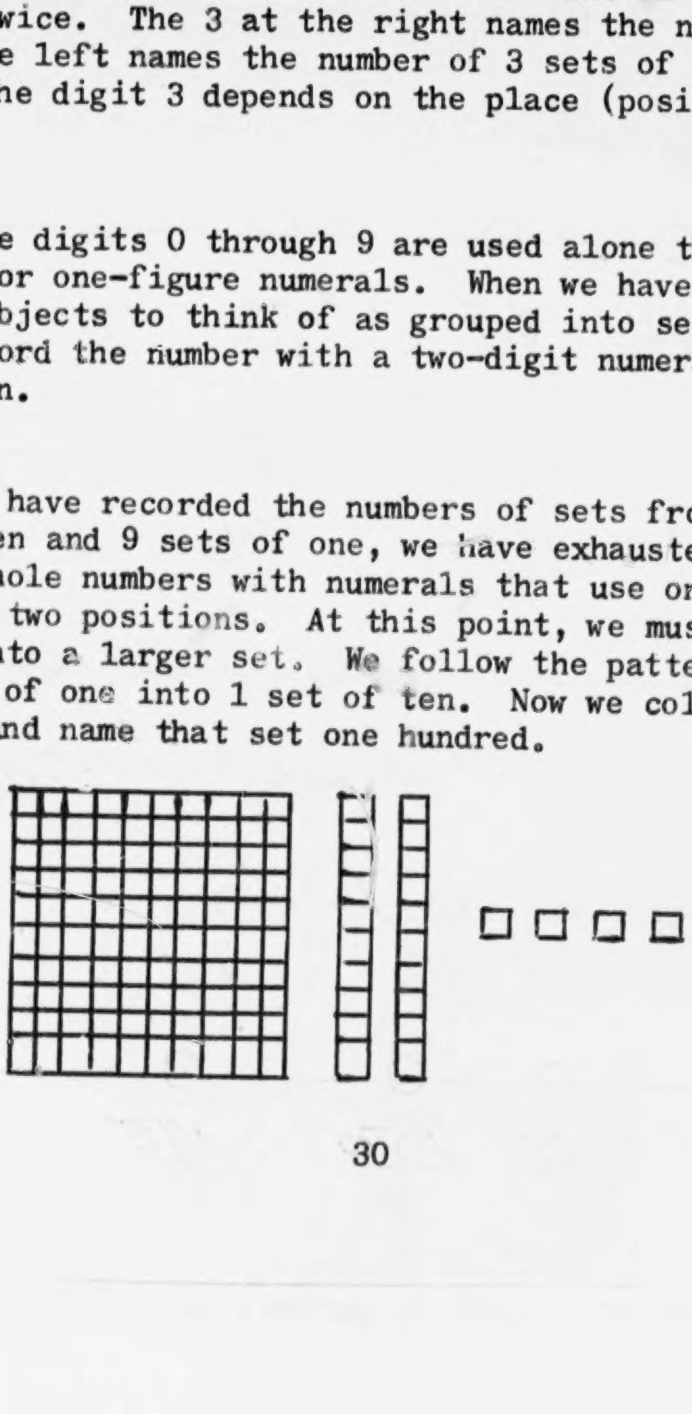
Illustration A:
xxxxxxxxxx ⑩
xxxxxxxxxx ⑩
xxxxxxxxxx ⑩
xxx 3

we might ask, "How many sets, each having ten members, can be formed?" We see that there are 3 sets of ten and 3 left over. The importance of place-value is evident when we attempt to write the numeral for the arrangement of objects (x's) in illustration A.

To name the number of any set of objects in our base-ten system of numeration, we use ten different basic symbols (sometimes called digits). These symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In decimal (base-ten) place-value numeration the number of objects in Illustration A above is expressed by the numeral 33; that is, there are 3 sets of ten and 3 sets of one (or one set of 3) members. Note that in the numeral 33 the digit 3 is used twice. The 3 at the right names the number of 3 sets of one; the 3 at the left names the number of 3 sets of ten. The number represented by the digit 3 depends on the place (position) it occupies in the numeral.

When the digits 0 through 9 are used alone they are called one-place, one-digit, or one-figure numerals. When we have a sufficiently large number of objects to think of as grouped into sets of ten and sets of one, we record the number with a two-digit numeral, as in the preceding illustration.

When we have recorded the numbers of sets from the empty set through 9 sets of ten and 9 sets of one, we have exhausted the possibilities of recording whole numbers with numerals that use only ten different symbols and two positions. At this point, we must think of objects as collected into a larger set. We follow the pattern we began by collecting 10 sets of one into 1 set of ten. Now we collect 10 sets of ten into 1 set and name that set one hundred.



How many sheets of 100 stamps? strips of ten stamps? single stamps?
124 means ? hundreds, ? tens, ? ones.

We follow a similar pattern in dealing with larger numbers of objects, grouping them into sets of ten hundreds (one thousand), ten thousands, ten ten-thousands (one hundred thousands). Each time we must use another place to the left in the numeral that records the number. For example:

a.
$$\begin{array}{r} 9 \\ +1 \\ \hline 10 \end{array} \qquad \begin{array}{r} 99 \\ +\ 1 \\ \hline 100 \end{array} \qquad \begin{array}{r} 999 \\ +\ \ 1 \\ \hline 1000 \end{array} \qquad \begin{array}{r} 9,999 \\ +\ \ \ 1 \\ \hline 10,000 \end{array}$$

b.
$$4,357 = 4000 + 300 + 50 + 7 \qquad \begin{array}{r} 4000 \\ 300 \\ 50 \\ 7 \\ \hline 4357 \end{array}$$

An important principle in our numeration system is that of place-value. Place-value simply means that the number a digit represents depends on the place it occupies in the complete numeral.

It will not be an easy task for pupils at this grade level to write and read numerals for large numbers. If you make it clear to them that any three digits in the fourth, fifth, and sixth places respectively name the thousands, they will be aided in recognizing that reading such numerals is no more difficult than reading in the hundreds or units. Do not expect all of your pupils to master reading and writing numerals through six places, but all pupils should understand basic place-value concepts.

TEACHING PROCEDURE

Exploration. Review with your pupils the concept that each position occupied by a numeral written in the decimal system has a name. For example, on the chalkboard write the numeral 647.* Point out that this is a three-place numeral and that each digit has a name and represents a certain value. Ask, "Who would care to analyze the numeral 647 and tell us what it represents?" The responses should be similar to the following:

(1) The numeral 647 is read: six hundred forty-seven.
(2) The 6 in 647 means 600 or 6 hundreds; the 4 in 647 means 40 or 4 tens; and the 7 in 647 means 7 ones.

---

* Perhaps the teacher may wish to explore this type of analysis by starting with a two-place numeral such as 38 instead of a three-place numeral. In either case, use the analysis similar to that mentioned above.

Next, write the numeral 999 on the chalkboard. Lead the pupils to see that 999 is a three-place numeral even though the same digit is used in all three positions. Ask, "Who could analyze the numeral 999 and tell us what it really represents?" Some of the responses might be:

1. There are 9 hundreds, 9 tens, and 9 ones.
2. From left to right, the first 9 means 900; the second 9 means 90; and the third 9 means 9 ones.

On the chalkboard show the following illustration and ask the pupils to make an analysis of each of the numerals. Ask, "Why are the numerals 4,307 and 6 placed as they are?"

ten thousands	thousands	hundreds	tens	ones
3	8,	8	1	3
	4,	3	0	7
			4	9
1	2,	1	5	8
		6	2	0
				6
1	6,	2	0	0

Because 4,307 represents: 4 thousands, 3 hundreds, 0 tens, and 7 ones.

Because 6 represents 6 ones.

Next, ask, "Why are the numbers represented by each of the numerals easy to read?"

1. Because each column has a special position and special name.
2. The chart tells us the name for each position.

Extend the concept of the decimal system by the following illustration, suggesting that the pupils assist by filling in the blank spaces:

10 ones are the same as 1 ten

10 tens are the same as 1 hundred

10 hundreds are the same as 1 thousand

10 thousands are the same as 1 ten thousand.

For an analysis of four-place numerals, write the numeral 1,234 on the chalkboard. Ask, "How many hundreds does this numeral represent?" 12 Name two ways in which we can read the numeral 1,234."

1. One thousand, two hundred thirty-four
2. Twelve hundred thirty-four.

To help pupils to further analyze the numeral 1,234 and others, ask the class to write the numerals represented by each of the digits and their arrangement. Illustration A should serve as a model:

Illustrations:	A	B	C	D
Numeral:	1,234	2,341	3,412	4,123
Meaning:	1,000 200 30 4	2,000 300 40 1	3,000 400 10 2	4,000 100 20 3

Have the pupils take note that each illustration has a different arrangement of the digits 1,234. Thus, because of place-value, a difference in arrangement of the digits changes their meaning. Each example has the same digits but represents a different number.

Next, have the pupils fill in the missing numerals, for the following examples. Have them do these at the chalkboard.

2,000	5,000	8,000	7,000
600	800	300	200
90	20	00	00
5	0	4	0
2,695	5,820	8,304	7,200

SUMMARY OF BASIC IDEAS

1. The place-value scheme that the decimal system utilizes has a base of ten.

2. The importance of place-value is evident especially when we want to express large numbers.

3. The ten basic symbols used by our system of numeration are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

4. When analyzing place-value we find that 10 ones = 1 ten; that 10 tens = 1 hundred; that 10 hundreds = 1 thousand; and that 10 thousands = 1 ten thousand.

5. The arrangement of a given set of digits in the numeral may change its meaning (value).

Pupil Exercises. Provide each pupil with worksheet No. 6.

Briefly discuss examples 1, 2, and 3 with the pupils to make sure they understand the instructions. Examples 4, 5, and 6 may need additional discussion, especially example 6.

Interim Mathematics Unit - Grade 4
San Diego City Schools (1-65)

Worksheet No. 6

READ THE INSTRUCTIONS FOR EACH SET OF EXAMPLES.

1. WRITE THE NUMERAL THAT MEANS:

a. 6 hundreds, 0 tens, 0 ones 600

b. 6 hundreds, 6 tens, 0 ones 660

c. 6 hundreds, 6 tens, 6 ones 666

2. FILL IN EACH BLANK SPACE WITH THE CORRECT DIGIT.

a. 9 + 1 = 1 ten and 0 ones

b. 99 + 1 = 1 hundred, 0 tens, and 0 ones

c. 999 + 1 = 1 thousand, 0 hundreds, 0 tens, and 0 ones

3. ARRANGE THE FOLLOWING NUMERALS IN THE CORRECT ORDER FROM LEAST TO GREATEST:

a. 3,538 3,358 4,583 8,543 4,538

3,358 , 3,538 , 4,538 , 4,583 , 8,543

b. 8,756 8,568 8,765 8,687 8,658

8,568 , 8,658 , 8,687 , 8,756 , 8,765

4. LIST THE NUMERALS FOR THE EVEN NUMBERS BEGINNING WITH 6 AND ENDING WITH 22.

{ 6, 8, 10, 12, 14, 16, 18, 20, 22 }

5. LIST THE NUMERALS FOR THE ODD NUMBERS BEGINNING WITH 13 AND ENDING WITH 29.

{ 13, 15, 17, 19, 21, 23, 25, 27, 29 }

6. IN THE BLANK SPACES, WRITE THE MISSING NUMERALS WHICH COMPLETE EACH PATTERN.

a. 990; 1000; 1,010; 1,022 ; 1,030 ; 1,040 ; 1,050

b. 1,400; 1,500; 1,600; 1,700 ; 1,800 ; 1,900 ; 2,000

c. 1,107; 1,108; 1,109; 1,110 ; 1,111 ; 1,112 ; 1,113

d. 56, 51, 46, 41 , 36 , 31 , 26 , 21 , 16

e. 127, 136, 145 , 154, 163 , 172 , 181 , 190, 199

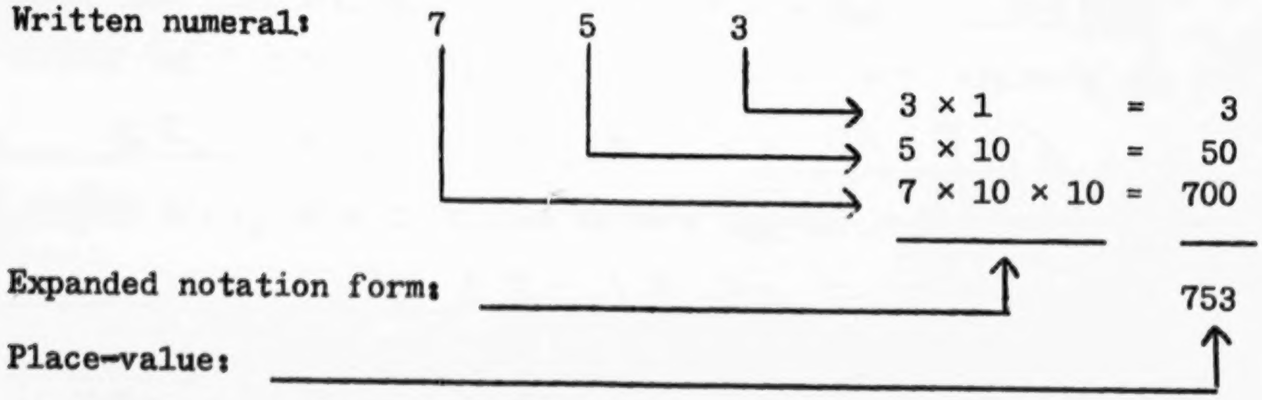
E. Expanded Notation

Objectives. (1) To develop the place-value meaning of a numeral by using the expanded notation form. (2) Provide opportunities to become familiar with a computing device (abacus) which illustrates place-value and permits place-value analysis.

Background Information for the Teacher. We have learned that the distinguishing feature of a positional system as opposed to a non-positional system of numeration is the use of basic numerals whose number values are determined by their position in a sequence of numerals. This is the concept of place-value.

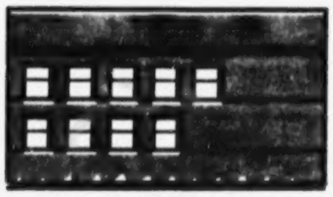
Our decimal system of numeration uses ten basic numerals, or digits (a derivation from the Latin word digitus, meaning finger). It is the use of the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 in a place-value scheme based on the powers of ten that we have our decimal numeration system.

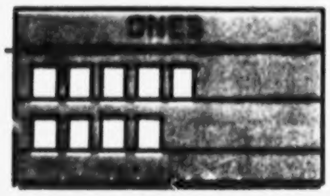
At this point, consider the numeral 753. The position occupied by each digit indicates (from right to left) the number of ones, tens, and hundreds. The position to which reference is made is an actual physical location relative to the ones position (see illustration below). From right to left, however, the number represented by the digit is ten times the value of the number represented by the digit which precedes it.



Further analysis of place-value can be achieved through the use of the place-value charts (synonym: pocket charts; place-value card holders) as shown below:

9 hundreds


9 tens

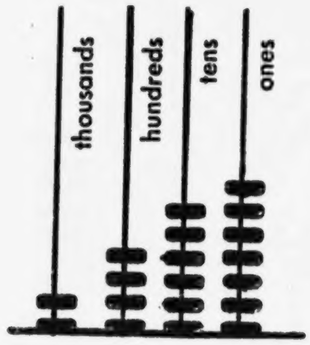


9 ones

	Expanded Notation						Numeral
	900	+	90	+	9	=	999
or	(9 × 100)	+	(9 × 10)	+	(9 × 1)	=	999
or	(9 × 10 × 10)	+	(9 × 10)	+	(9 × 1)	=	999

The abacus is a device which supplements the use of chalkboard illustrations regarding place-value and the pocket chart. An abacus is used in illustration A to analyze place-value.

Illustration A:



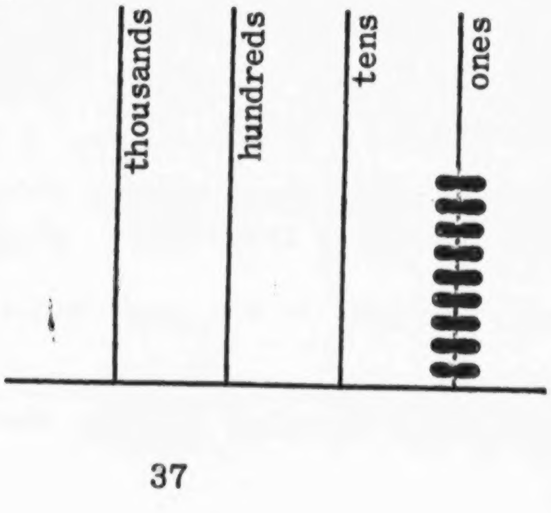
			1
		10	1
		10	1
	100	10	1
	100	10	1
1000	100	10	1
1000	100	10	1
2000	+400	+60	+7 = 2,467

From the illustrations listed on page 36, one can readily appreciate our positional system and the orderly way in which numbers can be analyzed and represented.

TEACHING PROCEDURE

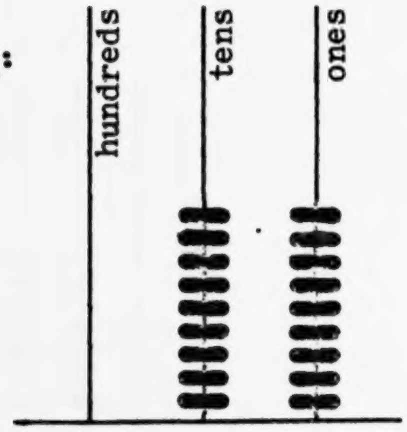
Exploration. A possible way to approach the teaching of expanded notation is to review with your pupils the concept of place-value. Next, introduce them to the abacus and emphasize its usefulness in terms of representing and analyzing numbers. On the chalkboard (or use a model if one is available) use the following, illustration B:

Illustration B:



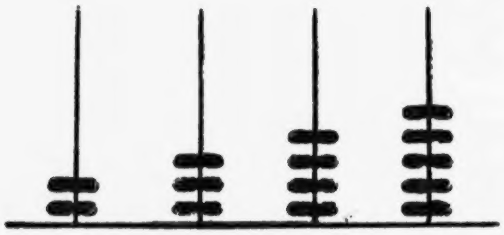
Ask, "How many ones are shown in illustration B?" 9 Next, illustrate an abacus which shows the following, illustration C:

Illustration C:



Ask, "How many tens and ones are shown on the abacus in illustration C?" 9 tens, 9 ones

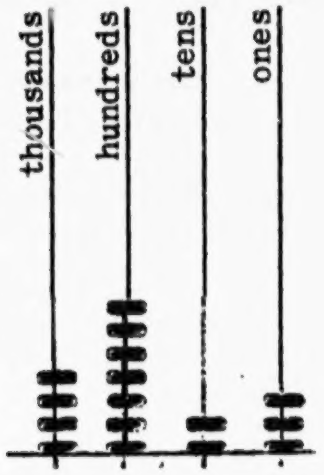
Now, illustrate an abacus which shows the following:



Ask the following questions:

1. How many ones are shown on the ones' wire? 5
2. How many tens are shown on the tens' wire? 4
   Write the numeral for 4 tens. 40
3. Each ten is equal to how many ones? 10
4. Then how many ones would it take to equal 4 tens? 40
5. How many hundreds are shown on the hundreds wire? 3
   Write the numeral for 3 hundreds. 300
6. Each hundred is equal to how many tens? 10
7. Then how many tens would it take to equal 3 hundreds? 30
   How many ones would it take? 300
8. How many thousands are shown on the thousands' wire? 2
   Write the numeral for 2 thousands. 2000
9. Each thousand is equal to how many hundreds? 10
10. Then how many hundreds would it take to equal 2 thousands? 20
    How many tens would it take? 200 How many ones? 2000

To analyze four-digit numerals, represent the following illustration on the chalkboard:



Ask the pupils to study the abacus illustration and have them write the numeral which is represented. 4,723 . Use the following questions and direction which are designed to help pupils analyze the place-value concept:

1. How many beads are on the wire in the thousands' place? 4 Do these beads represent the quantity 4 × 1,000? Yes What is the product of 4 × 1,000? 4,000

2. Write a numeral to show what the beads on the hundreds' wire represent? 700 What is 2 × 10? 20 What is 3 × 1? 3

3. Complete the following sentence represented by the abacus illustration:

   4,723 = 4,000 + 700 + 20 + 3

Next, indicate to the pupils that the abacus and the place-value idea permit us to read and express numbers in many different ways. On the chalkboard write the numeral 9,999. Ask, "Who could read this number in two ways?" Responses should be:

1. Nine thousand, nine hundred ninety-nine.
2. Ninety-nine hundred ninety-nine.

Now ask, "When we read it the second way, we are thinking of the 9 thousands as how many hundreds?" 90

Write the following numerals on the chalkboard and provide practice in reading and writing each example:

1,264 One thousand, two hundred sixty-four.
3,589 Three thousand, five hundred eighty-nine.

2,721 Two thousand, seven hundred twenty-one.

8,497 Eight thousand, four hundred ninety-seven.

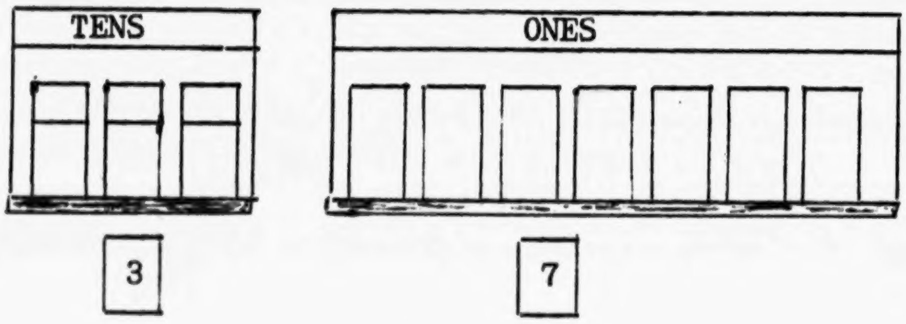
9,351 Nine thousand, three hundred fifty-one.

Have some pupils write these at their desks and ask others to write on the chalkboard. Check the correct use of the comma and the hyphen.

Next, on the chalk tray place a set of cards on which the ten digits have been written, that is:*

$\boxed{0}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$

Put a set of 37 colored paper strips in a place-value chart hanging above the chalk tray. Have a pupil count by ones to determine the number of the set of colored paper strips. After the pupil (or the class) has concluded that this is a slow method of finding the number of members in the set, have the pupil form as many sets of ten strips of colored paper as he can. Have him use paper clips or rubber bands to bind each set of ten strips together as in the illustration below:



Now ask someone to choose from the set of cards those which are needed to represent the number of sets of ten and sets of one, that is, cards $\boxed{3}$ and $\boxed{7}$. Emphasize to the class that the 7 digit occupies a place in the numeral called the ones' place and 3 digit occupies the place in the numeral called the tens' place.

Under the pocket chart, below the 3 and the 7 write these symbols:

$\boxed{3}$ $\boxed{7}$

3 × 10 7 × 1

---

*This activity can be modified depending on the maturity level and experience with the place-value concept.

Point out to the pupils that by adding the quantity represented by those two sets, we find the whole number we are seeking. For example:

$$\boxed{3} \qquad\qquad \boxed{7}$$

$$3 \times 10 = 30 \quad \text{and} \quad 7 \times 1 = 7$$

$$30 \quad + \quad 7 \quad = 37$$

At this time, indicate to the pupils that only two numbers can be added at a time. A standard symbol must be used to show that numbers are being "grouped" and should be added first. Thus, we use "( )," which we call parentheses. With this new symbol we can show the above example in the following form:

$$\boxed{3} \qquad\qquad \boxed{7}$$

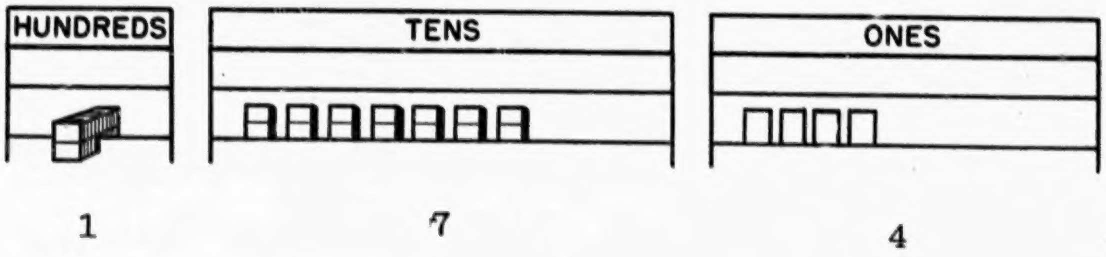
$$(3 \times 10) \quad \text{and} \quad (7 \times 1) =$$

$$30 \quad + \quad 7 \quad = \quad 37$$

Now, remind the pupils that 37 is the numeral which represents the expanded form 30 + 7.

Continue with three or four similar examples, using a set of twenty-nine strips, a set of fifty-three strips, and a set of eighty-nine strips.

Next, place seventeen sets of ten strips in the pocket chart. Have someone count the number of sets of ten to determine the number of strips in this set. Lead the class to conclude that, although this is much quicker than counting by ones, there is a more efficient method that can be used, that is, group 10 sets of ten to form 1 set of one hundred. Thus, we now have (See illustration) the numeral 174.



Now, ask, "What does each digit in the numeral represent?" Responses should be similar to:

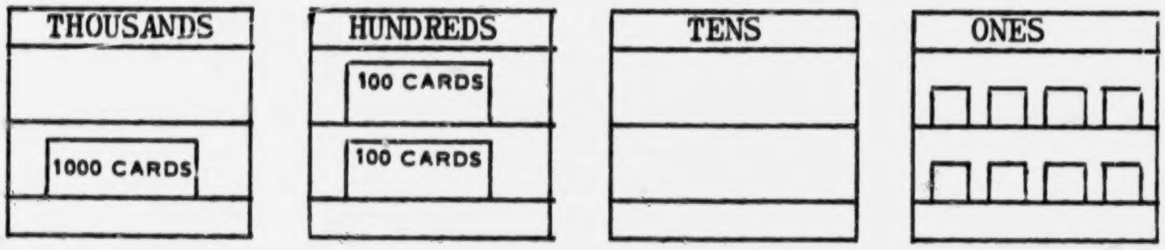
(1) The digit one (1) represents one set of one hundred.
(2) The digit seven (7) represents seven sets of ten.
(3) The digit four (4) represents four sets of one.

Again explain that other symbols can be used to show the number of hundreds, tens, ones, that is:

$$\overset{\textcircled{1}}{(1 \times 100)} + \overset{\textcircled{7}}{(7 \times 10)} + \overset{\textcircled{4}}{(4 \times 1)}$$

$$100 + 70 + 4 = 174$$

When dealing with thousands, use an example similar to that of the one below. Again explain that by using expanded notation, we can find the number as shown in the illustration below. Have the pupils note the empty set of TENS and the way it appears in the expanded notation form. For example:



$$(1 \times 1000) + (2 \times 100) + (0 \times 10) + (8 \times 1)$$

$$1{,}000 + 200 + 0 + 8 = 1{,}208$$

An example which the pupils will need to analyze might be number 555. Write the number 555 on the chalkboard. Have a pupil write the place-value meaning of this symbol under the given numeral. For example:

$$555$$

$$(5 \times 100) + (5 \times 10) + (5 \times 1)$$

$$500 + 50 + 5$$

Ask the pupils to respond to the following question: "Who can make a general statement about the digit 5 in each position?" The responses should be similar to the following:

1. The digit 5 on the right indicates that there are 5 sets, each containing one member.
2. The digit 5 in the middle also indicates five sets, each of which contains one member.

3. The digit 5 at the left also indicates five sets, each of which contains one member.

Guide the pupils to conclude that as we move from right to left, each position in the numeral is associated with sets ten times as large as those associated with the previous position.

Present other numerals in which a single digit is repeated three times. Have pupils come to the chalkboard to write the place-value meaning of each symbol as well as the expanded notation. Guide your pupils to generalize that a single digit can represent many different numbers, depending upon the position the digit holds in a numeral.

Write numerals containing the digit 0 in various positions such as 560 and 506. In each instance ask pupils to indicate the place-value meaning of the decimal numeral and to write the expanded notation for the given numeral. For example:

$$\begin{aligned} 560 &= (5 \times 100) + (6 \times 10) + (0 \times 1) \\ &= 500 + 60 + 0 \end{aligned}$$

$$\begin{aligned} 506 &= (5 \times 100) + (0 \times 10) + (6 \times 1) \\ &= 500 + 0 + 6 \end{aligned}$$

Pupil Exercises. Provide each pupil with worksheet No. 7.

Since many pupils are not familiar with this type of analysis and notation, it is suggested that similar examples be written on the chalkboard for group practice prior to doing the worksheet independently.

Worksheet No. 7

STUDY EXAMPLES 1 AND 6 BEFORE DOING EACH SET OF EXERCISES.

	Numeral	Meaning	Expanded Notation
1.	57	5 tens and 7 ones	50 + 7
2.	83	8 tens and 3 ones	80 + 3
3.	105	1 hundred, 0 tens and 5 ones	100 + 0 + 5
4.	476	4 hundreds, 7 tens and 6 ones	400 + 70 + 6
5.	980	9 hundreds, 8 tens and 0 ones	900 + 80 + 0

COMPLETE THE FOLLOWING:

6.	23	(2 × 10) + (3 × 1)	20 + 3
7.	32	(3 × 10) + (2 × 1)	30 + 2
8.	40	(4 × 10) + (0 × 1)	40+0
9.	321	(3 × 100) + (2 × 10) + (1 × 1)	300+20+1
10.	207	(2 × 100) + (0 × 10) + (7 × 1)	200+0+7
11.	305	(3 × 100) + (0 × 10) + (5 × 1)	300 + 0 + 5
12.	300	(3 × 100) + (0 × 10) + (0 × 1)	300 + 0 + 0
13.	33	(3 × 10) + (3 × 1)	30 + 3
14.	333	(3 × 100) + (3 × 10) + (3 × 1)	300 + 30 + 3
15.	3333	(3 X 1000)+(3 X 100)+(3 X 10)+(3 X 1)	3000+300+30+3

F. Renaming Numbers (Thinking about numbers in different ways)

Objectives. (1) To help pupils understand that the structure of our numeration system permits thinking about and naming numbers in several ways. (2) To develop skill in renaming numbers which is vital to understanding the "regrouping" idea used in the basic operations: addition, subtraction, multiplication, and division.

Background Information for the Teacher. The idea that a number has many names was introduced in the section entitled "Number-Numeral Distinction," page 21. With this understanding of place-value and the use of such devices as place-value charts and the abacus, it is now possible to develop this concept in greater depth.

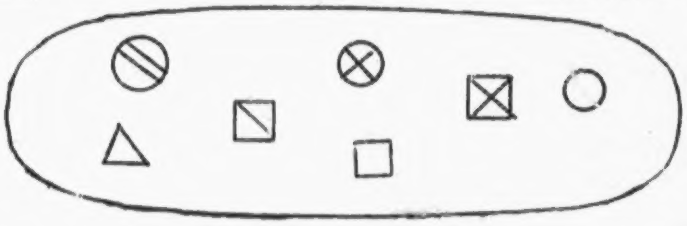
Consider the number 7 as an example; this can be shown as: 3 and 4, seven, (8 - 1), (1 × 7), (2 + 2 + 3), VII, etc. When dealing with a two-digit numeral such as 58, the following possibilities may be considered: 5 tens and 8 ones, 4 tens and 18 ones, 40 ones plus 18 ones, 38 and 20, etc. Three-digit, four-digit, and other numbers lend themselves to the same pattern. For example, the number 4,605 can be renamed in many ways using ones, tens, hundreds, and thousands:

4,605 = 4,605 ones
4,605 = 46 hundreds + 0 tens + 5 ones
4,605 = 4 thousands + 6 hundreds + 5 ones
4,605 = 460 tens + 5 ones
4,605 = 4,000 + 600 + 5
4,605 = 4,600 + 5
4,605 = 4,500 + 100 + 5
4,605 = 4,500 + 90 + 15

The last two names for 4,605 are especially useful in the understanding of addition and subtraction.

TEACHING PROCEDURE

Exploration. Show the following illustration on the chalkboard:



Next, ask someone to come to the chalkboard and write the numeral which represents the number of objects in the illustration. Of course, the numeral 7 names the number of objects represented. Ask the pupils if they will accept the following as names for the same illustration, also:

Seven
6 + 1
8 - 1

5 + 2
5 ones + 2 ones
VII

This should lead to a discussion and provide an opportunity for three-digit numbers. For example, on the chalkboard represent the following chart and numerals and have pupils write different names for the same number. Responses might be:

7	12	16	58
seven	10 and 2	8 and 8	5 tens + 8 ones
VII	6 times 2	sixteen	4 tens + 18 ones
5 and 2	12 ones	1 ten and 6 ones	38 and 20
7 ones	dozen	XVI	50 and 8
(6+1)	(13-1)	(2x8)	(20+20+10+8)

Next, use place-value charts, an abacus, bundles of sticks, and/or other devices to explore ways of naming numbers with three-digit numerals. Emphasis should be placed on finding more than one name for these numbers. The number 243, for example, may be expressed as:

243 = two hundreds, four tens, and three ones
243 = two hundreds, three tens, and thirteen ones
243 = one hundred, thirteen tens, and thirteen ones

After pupils become familiar with different ways of expressing numbers, ask them to find another name for such forms as: seven tens and eleven ones: 81; twenty-two tens and thirteen ones: 233; etc.

Inclusion of such groupings as three hundreds, twelve tens, and eighteen ones: 438; etc., is advisable because the understanding of these ideas are necessary for regrouping in addition and subtraction.

It is important that pupils develop an understanding of these ideas, especially through the use of manipulative materials. A great deal of group discussion and exploration of the numerous possibilities of different names for the same number should be undertaken before individual written exercises are attempted.

If more examples are needed for the development of a better understanding, use the following examples and have pupils come to the chalkboard to fill in the blank spaces or record their responses as they are given:

225 = 2 hundreds + 2 tens + 5 ones
225 = 22 tens + 5 ones
225 = 225 ones
225 = 1 hundred + 12 tens + 5 ones
225 = 2 hundreds + 1 ten + 15 ones
*225 = 200 + 20 + 5
*225 = 200 + 15 + 10

* Answers will vary.

Pupil Exercises. Provide each pupil with worksheet No. 8.

With the more able group, limit the explanation of this worksheet. However, with the less able group, offer as much explanation as you feel is necessary, especially with the examples requiring spelling.

WRITE "true" OR "false" IN EACH BLANK SPACE.

1. 34 may be renamed as 3 tens + 4 ones. true

2. 34 may be renamed as 4 tens + 3 ones. false

3. 34 may be renamed as 1 ten + 24 ones. true

4. 365 may be renamed as 3 hundreds + 5 tens + 15 ones. true

5. 365 may be renamed as 3 hundreds + 60 tens + 5 ones. false

WRITE THE CORRECT NUMERAL IN EACH BLANK SPACE.

6. Six hundreds + five tens + three ones = 653

7. Three hundreds + twelve tens + eight ones = 428

8. 4 hundreds + 17 tens + 8 ones = 578

9. 7 hundreds + 5 tens + 16 ones = 766

10. 8 thousands + 6 hundreds + 3 tens + 12 ones = 8,642

COMPLETE THE FOLLOWING SENTENCES:

11. 354 = 3 hundreds + 5 tens + 4 ones

12. 354 = 2 hundreds + 14 tens + 14 ones

13. 505 = 4 hundreds + 0 tens + 105 ones

*14. Name the number 836 in three different ways:

836 = 8 hundreds + 3 tens + 6 ones
836 = 7 hundreds + 13 tens + 6 ones
836 = 6 hundreds + 23 tens + 6 ones

---

* Answers will vary.

G. Number Sentences

Objectives. (1) To help pupils understand the need for number sentences and (2) to develop skill in describing a problem in their own words and then to state the relationship by use of a mathematical sentence.

Exploration. During the language arts period, pupils use words to write sentences which express ideas, feelings, descriptions, etc. In mathematics, numerals and various symbols are used to express ideas, that is, a number sentence (synonym: mathematical sentence; mathematical equation; equation).

Number sentences tell something about numbers and the action related to those numbers. For example, $6 + 7 = 13$ indicates that the numbers 6 and 7 are being operated on by addition; the result being 13. In other words, it is a sentence about 6, 7, 13 and the operation of addition.

When writing number sentences such as $6 + 7 = 13$, we are expressing the idea that both, $6 + 7$ and 13 are different names for the same number. Other names for 13 may be: $9 + 4$; $18 - 5$; $(2 \times 6) + 1$; etc.

Number sentences may be true or false. Examples of different kinds of sentences are shown below:

True sentences: $8 + 7 = 15$; $4 \times 5 = 20$; $(3 + 3) \times 2 = 12$

False sentences: $3 + 7 = 11$; $11 \times 2 = 23$; $(3 + 4) \times 0 = 7$

Open sentences: $4 + 7 = \square$; $19 - \square = 6$; $\square \times 2 = 6$
(Neither true nor false)

When a pupil learns to describe a problem in his own words and then states the relationship in a number sentence form, he probably has a sound approach to problem solving.

TEACHING PROCEDURE

Exploration. Write the following sentence on the chalkboard:

> "Seven girls and fourteen boys spent an afternoon at the San Diego Zoo studying lions and tigers. How many children were involved in this study?"

Next, indicate to the pupils that words and sentences were used to describe an idea involving sets of girls, boys, numbers, etc. To solve the problem, we would write:

$$7 + 14 = 21$$

It should be emphasized that when combining sets we do not add the objects (boys and girls); we add the numbers associated with the sets.

Point out that numerals and other symbols can be used to express number ideas; this method is the number sentence. Ask, "How can we express this story and solve the problem using a number sentence?" Response might be:

7 + 14 = 21

14 + 7 = □

◇ = 14 + 7

7 + 14 = ?

7 + 14 = n

□ = 7 + 14

Encourage a variety of responses, many of which will serve as an introduction to number sentences. Next, indicate to the pupils that sometimes more than two numbers are mentioned in a word sentence. For example, write the following sentence on the chalkboard:

> "Three boys went skin diving near the La Jolla Cove. One boy speared 3 fish, another speared 5 fish, and the third boy speared only 1 fish. How many fish were caught by the three boys?"

Now ask, "How can we state this problem in number sentence form?" Some responses might be:

3 + 5 + 1 = □

3 + 5 + 1 = 9

If pupils do not give examples using parentheses such as (3 + 5) + 1 = 9 or 3 + (5 + 1) = 9, review with them how, when, and why they are used. For example, the teacher might point out that since only two numbers can be added (this also applies to subtraction, multiplication, and division) at any one time, parentheses are used to indicate which numbers are being operated on first. Parentheses are important to the understanding of the "grouping principle" or associative principle of addition and multiplication. (The phrase associative principle need not be used with fourth-grade pupils.)

Give pupils an opportunity to work on the chalkboard or have them copy and practice some of the following examples of number sentences: *

	(a)	(b)	(c)
1.	4 + 7 = 11	3 + 4 + 7 = 14	(2 + 6) + 10 = 18
2.	7 + 12 = 19	$\boxed{12}$ + 5 + 10 = 27	(12 − 4) + 7 = $\boxed{15}$

---

*It is common practice to write the missing numeral in the blank space or within the frame in order to complete each number sentence. However, whenever the pupil is requested to replace the frame with the missing numeral, for example, □ + 14 = 20, he should write his response as: 6 + 14 = 20 (Do not require him to rewrite the example with the frame and then have him write the missing numeral in this frame.)

3. $15 - \boxed{3} = 12$ $17 + \boxed{3} + 10 = 30$ $(\boxed{10} + 0) + 5 = 15$

4. $19 - 7 = \underline{12}$ $4 + 10 + \underline{6} = 20$ $\boxed{9} + (7 + 3) = 19$

After the examples are completed by most pupils, call on them to give their solution to each of the examples. This can be a good learning situation for the pupils and an opportunity for the teacher to emphasize certain important points, that is: (a) restating problems as number sentences, (b) in solving number sentences you may only work with two numbers at a time, (c) number sentences can be shown using different forms, and (d) the operation in parenthesis should be completed first.

In Grade 4, Modern Arithmetic Through Discovery, the inequality symbol, < (meaning "less than"), is introduced. The inequality concept and symbol is very useful, especially in the development of division.

Before introducing the symbol for "less than," indicate to the pupils that, for example, "5 × 7" and "35" are two names for the same number. The relation symbol, = , used in the number sentence 5 × 7 = 35, shows this relationship. Since 5 × 7 "equals+ 35, it must be "less than," for example, 36. To show this sentence, write the following inequality sentence on the chalkboard:

$$5 \times 7 < 36$$

Point out that the sentence is read "5 times 7 is less than 36."

Write several examples on the chalkboard such as illustrations (a), (b), and (c) shown below:

(a)	(b)	(c)
$7 \times 9 < 65$	$8 \times 6 < 65$	$\square \times 4 < 21$

Regarding each illustration ask, "In illustration (a) are there 7 nines in 65? Yes Is the sentence true? Yes In illustration (b) what is the largest number of 6's that are contained in 49? 8 Is the sentence true? Yes In illustration (c) what is the numeral for the largest number you can use to replace the frame and make the sentence true?" 5

(In the sentence $\square \times 4 < 21$, the numerals 0, 1, 2, 3, and 4 can also be used as replacements for the frame with the result that the numbers named will make a true sentence. But, only 5 is the largest number that will still yield a true sentence.)

Use other examples to develop the inequality concept. After a variety of illustrations, including the number line, write the following on the chalkboard. Have pupils copy each example and have them replace the frame (or write within the frame) with the numeral for the largest number that will make the sentence true.

	(a)	(b)
1.	$\boxed{4} \times 2 < 9$	$\boxed{6} \times 9 < 58$
2.	$\boxed{3} \times 5 < 17$	$\boxed{5} \times 9 < 50$
3.	$\boxed{8} \times 6 < 53$	$\boxed{6} \times 7 < 47$
4.	$\boxed{5} \times 8 < 47$	$\boxed{8} \times 4 < 35$
5.	$\boxed{9} \times 9 < 88$	$\boxed{9} \times 7 < 69$

Pupil Exercises. Provide each pupil with worksheet No. 9.

Unless the group is advanced, read examples 11 through 14 with the pupils and have them respond orally. Then have them do the exercises independently.

Worksheet No. 9

WRITE THE MISSING NUMERAL FOR EACH EXAMPLE IN THE FRAME.

	(a)	(b)
1.	$56 + 4 = \boxed{60}$	$9 + 6 + 7 = \boxed{22}$
2.	$26 + 5 = \boxed{31}$	$7 + 9 + 6 = \boxed{22}$
3.	$3 + 28 = \boxed{31}$	$\boxed{6} + 14 + 10 = 30$
4.	$\boxed{17} + 33 = 50$	$14 + 10 + 6 = \boxed{30}$
5.	$48 + 7 = \boxed{55}$	$7 + 14 + 7 = \boxed{28}$

DO THE WORK IN PARENTHESES FIRST BEFORE COMPLETING EACH EXAMPLE.

6.	$(14 + 16) + 15 = \boxed{45}$	$(17 + 33) + 14 = \boxed{64}$
7.	$(11 + 19) + 12 = \boxed{42}$	$10 + (19 + 11) = \boxed{40}$
8.	$(44 + 26) + 10 = \boxed{80}$	$7 + (7 + 33) = \boxed{47}$
9.	$(0 + 0) + 99 = \boxed{99}$	$10 + (1 + 0) = \boxed{11}$
10.	$(18 - 8) + 10 = \boxed{20}$	$(24 - 14) + 8 = \boxed{18}$

WRITE A COMPLETE NUMBER SENTENCE FOR EACH PROBLEM. *

11. Mary collected 88 shells at Imperial Beach. She also had a collection of 32 shells at home. How many shells does she now own?

$88 + 32 = 120$

12. The distance from Jack's home to Mission Valley shopping center is 23 miles. How many miles will Jack travel if he makes a round trip?

$23 \times 2 = 46$ or $23 + 23 = 46$

13. Bob and Joe collected matchbook covers. Bob had a set of 175 different matchbook covers and Joe had a set of 123. How many matchbook covers were in the two collections?

$175 + 123 = 298$

14. Mark traveled from his home to Sea World which is a distance of 9 miles. From there he went to Pacific Beach which is a distance of 4 miles. Then he traveled to La Jolla Cove, a distance of 5 miles. How many miles did he travel altogether?

$(9 + 4) + 5 = 18$

COMPLETE THE PATTERN.

15. 28, 36, 44, 52, 60, 68, 76, 84, 92

---

*Number sentences will vary.

CHAPTER III

OPERATIONS

The developmental ideas in this chapter are devoted to an analysis of the mathematical principles used in basic operations on numbers. The following are considered: (a) analyzing number relationships in a problem defined by a mathematical sentence, (b) deciding on what operation to use, and (c) performing the operation on the numbers.

Pupils who have developed skill in solving problems frequently use many procedures or steps in a different order in solving a variety of problems.

No doubt, pupils have had many experiences with addition, subtraction, multiplication, and division. However, they need to consider the nature of these operations carefully to expand and reinforce the necessary concepts.

It is assumed that this will be the first time pupils will have been formally exposed to such principles as commutative principle of addition, commutative principle of multiplication, associative principle of addition, associative principle of multiplication, and the distributive principle of multiplication over addition.

A. Addition and Subtraction

1. Adding and Subtracting: Tens and Ones

Objective. To develop an understanding of the common algorisms used in addition and subtraction.

Background Information for the Teacher. The development of the meaning basic to the techniques of addition and subtraction is of great importance and should not be slighted. An analysis of the techniques should be given emphasis since they provide a means of understanding the structure of the operations.

If the expression $42 + 37 = \square$ were written on the chalkboard, the sum could be written directly as 79 as in Illustration (a) below. However, if 67 and 58 are the numbers and the operation is addition, the result is 11 tens and 15 ones. This sum, obviously, cannot be written directly as a numeral but may be thought of as 1 hundred, 2 tens, and 5 ones. Then the numeral is written as 125 as in Illustration (b) on the following page:

Illustration (a)

$$\begin{array}{r} 42 \\ +\ \underline{37} \\ 79 \end{array}$$

2

Illustration (b)*

$$\begin{array}{rcll} 67 & = & 6\text{ tens}, & 7\text{ ones} \\ +\,58 & = & 5\text{ tens}, & 8\text{ ones} \\ \hline & & 11\text{ tens}, & 15\text{ ones} = 125 \end{array}$$

Similarly, when the operation of subtraction is used with numbers whose numerals have more than one digit, the naming of the difference requires careful thinking. For example, if 49 and 23 are the numbers and the operation is subtraction, the difference (unknown addend) can be written directly as the numberal 26 [Illustration (c)]. However, if 32 and 17 are the numbers operated on, the 32 must be thought of as 2 tens and 12 ones. This makes it convenient to subtract 1 ten and 7 ones; the result is then written as 15 [Illustration (d)].

Illustration (c)

$$\begin{array}{r} 49 \\ -\,23 \\ \hline 26 \end{array}$$

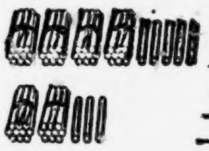
Illustration (d)*

$$\begin{array}{rcll} 32 & = & 2\text{ tens}, & 12\text{ ones} \\ -\,17 & = & 1\text{ ten}, & 7\text{ ones} \\ \hline & & 1\text{ ten}, & 5\text{ ones} = 15 \end{array}$$

Performing addition and subtraction in this way does not mean that we add or subtract from left to right. In regrouping for addition and subtraction it is convenient to use two numerals rather than one to express a given number. This procedure strengthens the place-value concept.

TEACHING PROCEDURE

Exploration. On a large flannel board display an arrangement of bundles of sticks on strips of colored paper (a chalkboard arrangement of x's will also serve as an illustration), such as:



$$\begin{array}{rcl} 45 & = & 40 + 5 \\ +\,23 & = & 20 + 3 \\ \hline \longrightarrow & & + \end{array}$$

Have the pupils study the arrangement so that they can respond to the following questions:

(1) "How many sticks are in the first row? 45

Rename the number 45." 4 tens and five ones, or 44+1, 43+2, etc.

---

*The following form may be used for illustrations (b) and (d) above:

(b)

$$\begin{array}{rcl} 67 & = & 6\text{ tens} + 7\text{ ones} \\ +\,58 & = & 5\text{ tens} + 8\text{ ones} \\ \hline & & 11\text{ tens} + 15\text{ ones} = 125 \end{array}$$

(d)

$$\begin{array}{rcl} 32 & = & 2\text{ tens} + 12\text{ ones} \\ -\,17 & = & (1\text{ ten} + 7\text{ ones}) \\ \hline & & 1\text{ ten} + 5\text{ ones} = 15 \end{array}$$

(2) What is the number in the second row? 23

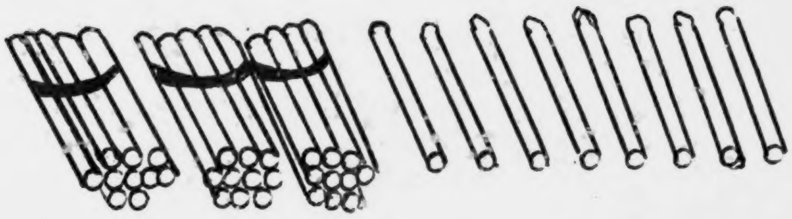
This is how many tens and how many ones? 2 tens and 3 ones.

(3) What is the "shortest" name for 40 + 20? 60

The "shortest name for 5 + 3? 8

(4) What is the sum of 60 + 8? 68

After exploring examples such as the one above, introduce the pupils to regrouping to find the sum. For example, on the flannel board or chalkboard, represent the following: *



38 = 3 tens, 8 ones

+ 25 = 2 tens, 5 ones

→ 5 tens, 13 ones, or □

Point out to the pupils that sometimes when we add, the sum of the ones is more than 9. Then we have to regroup the ones into tens and ones. Ask the pupils to study the above example so that they may answer the following questions:

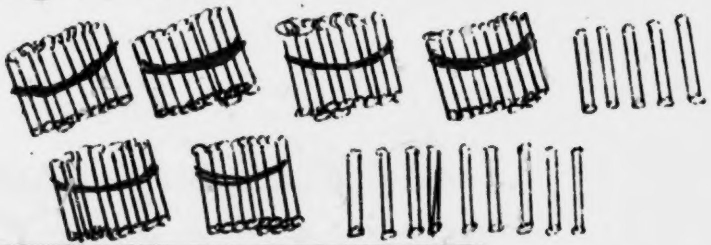
(1) Examine the illustration on the chalkboard. The example using the bundles of sticks shows that we are to add 38 and 25. What is another name for 38? 3 tens and 8 ones. What is another name for 25? 2 tens and 5 ones. What is the sum as shown by the illustration? 5 tens and 13 ones.

(2) Can we regroup 13 ones? Yes, 1 ten and 3 ones.

(3) If we add the 1 ten to the 5 tens, how many tens and how many ones will we have? 6 tens and 3 ones

(4) What is the numeral for 6 tens and 3 ones? 63

Next, use the following example to illustrate the nature of renaming the sum:



45 = ____ tens, ____ ones

+ 29 = ____ tens, ____ ones

→ 6 tens, 14 ones, or ____

---

*Slower pupils will need to work with concrete objects. Milk straws or paper strips may be fastened to the chalkboard with adhesive to illustrate the numbers to be added. A pupil may write the appropriate numerals on the chalkboard beside the straws or strips, while other pupils write on their papers. Take time to develop this regrouping idea. Many experiences will be necessary before all pupils understand the concept.

Have the pupils study this illustration so that they may answer the following questions:

(1) "What numerals will represent the illustration and rename 45?" 4 tens and 5 ones.

(2) "What numerals will represent the illustration and rename 29?" 2 tens and 9 ones.

(3) "What is the sum of the ones? 14 The sum of the tens? 6 Can we regroup the 14 ones as 1 ten and 4 ones? Yes How many tens and how many ones do we have now? 7 tens and 4 ones. What names 7 tens and 4 ones?" 74

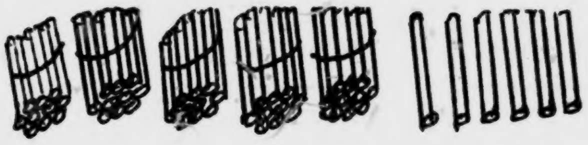
Emphasize the importance of understanding the conventional algorithm by using examples such as:

Conventional Form	Expanded Forms		
49 + 37	4 tens, 9 ones 3 tens, 7 ones 7 tens, 16 ones	or	40 + 9 30 + 7 70 + 16
	or		or
	8 tens, 6 ones		80 + 6 = 86

While some pupils come to the chalkboard to work examples for practice, have the other pupils work some of the following at their desk, using one of the expanded notation forms:

(a)	(b)	(c)	(d)	(e)
56	38	57	71	84
+27	+46	+29	+19	+7

To develop the idea of subtraction using the regrouping concept, use the following illustration:



$$\begin{array}{r} 56 = 50 + 6 \\ \underline{-\ 24} = \underline{20 + 4} \\ \longrightarrow \quad + \end{array}$$

Have the pupils study the illustration so that they may answer the following questions and be guided to discover a relationship between addition and subtraction. Ask the following:

(1) "Can we subtract 24 from 56 by subtracting the tens, subtracting the ones, and then putting the answers together?" Yes

(2) "Let's investigate. What is 50 - 20? 30 What is 6 - 4? 2 What is 30 + 2? 32 Then what is 56 - 24?" 32

Next, write the following problem on the chalkboard:

> "Several boys put 63 chairs in the auditorium where the fourth grade planned to give a performance. Later, someone removed 26 chairs. How many chairs are left in the auditorium for the performance?"

Ask, "Who can write a number sentence for this problem?" Accept any of the following:

(a) $63 - 26 = \square$

(b) $63 - 26 = \underline{\quad\quad}$

(c) $63 - 26 = \triangle$

Now, write the expanded form for the above number sentence so that the pupils will see how the structure of the expanded form permits an analysis of the regrouping concept with respect to subtraction. For example:

$$\begin{array}{rcl} 63 & = & 6 \text{ tens, } 3 \text{ ones} \\ -\underline{26} & = & \underline{2 \text{ tens, } 6 \text{ ones}} \end{array}$$

After pupils have studied the illustration, ask, "Is it possible to subtract 6 ones from 3 ones and get a whole number? no What must we do to perform this operation?" (Guide the pupils to use the regrouping concept, renaming 63 as 5 tens and 13 ones.) * For example:

$$\begin{array}{rcccl} 63 & = & 6 \text{ tens, } 3 \text{ ones} & = & \underline{5} \text{ tens, } \underline{13} \text{ ones} \\ -\ 26 & = & 2 \text{ tens, } 6 \text{ ones} & = & \underline{2} \text{ tens, } \underline{6} \text{ ones} \\ \hline & & & & \end{array}$$

Next, having renamed 63, show the following, completing illustration (a), then showing the form [illustration (b)]:

$$\begin{array}{rcccl} & & \text{(a)} & & \text{(b)} \\ 63 & = & 5 \text{ tens, } 13 \text{ ones} & = & 50 + 13 \\ -\ 26 & = & 2 \text{ tens, } \ \ 6 \text{ ones} & = & \underline{20 + \ \ 6} \\ \underline{\quad\quad} & & \overline{3 \text{ tens, } \ \ 7 \text{ ones}} & = & 30 + \ \ 7 = 37 \end{array}$$

---

* The teacher may wish to illustrate the minuend 63 with 6 bundles of ten counting sticks and 3 single sticks fastened to the chalkboard or shown in the place-value chart. Let the pupils suggest untying one of the bundles of ten so that there will be 5 tens and 13 ones. Then, let a pupil remove six of the single sticks and 2 of the bundles of ten to illustrate subtracting 26 from 63.

The teacher using this unit is encouraged to illustrate and work several examples with the pupils similar to those shown above. When the teacher feels that the concept has been thoroughly developed and understood, write the following on the chalkboard:

(a)	(b)	(c)	(d)	(e)
81	71	84	90	92
- 57	- 23	- 47	- 53	- 53

While some pupils come to the chalkboard to work examples for practice, have the other pupils work some of the examples at the desks. Have them proceed from the conventional form to the expanded forms. When sufficient mastery has taken place, proceed with the pupil exercises.

Pupil Exercises. Provide each pupil with worksheet No. 10.

A brief explanation of this worksheet will be necessary since a variety of forms are used. Remind the pupils to examine each example; some are addition and some are subtraction.

Interim Mathematics Unit - Grade 4 Worksheet No. 10
San Diego City Schools (1-65)

STUDY EACH EXAMPLE CAREFULLY. WRITE THE MISSING NUMERALS IN EACH BLANK SPACE OR FRAME.

1. 66 = 6 tens, 6 ones = 60 + 6

+ 27 = 2 tens, 7 ones = $\boxed{20}$ + 7

$\boxed{8}$ tens, $\boxed{13}$ ones = 80 + 13 = 93

2. 48 = 4 tens, 8 ones = 40 + 8

+ 29 = $\boxed{2}$ tens, 9 ones = 20 + 9

6 tens, 17 ones = 60 + 17 = 77

3. 63 = 6 tens, 3 ones = 5 tens, 13 ones = 50 + 13

− 29 = $\boxed{2}$ tens, 9 ones = 2 tens, $\boxed{9}$ ones = − (20 + 9)

3 tens, 4 ones = 30 + 4 = 34

4. 81 = $\boxed{8}$ tens, 1 one = 7 tens, 11 ones = 70 + 11

− 67 = 6 tens, 7 ones = $\boxed{6}$ tens, 7 ones = − (60 + 7)

$\boxed{1}$ ten, 4 ones = 10 + 4 = 14

(a) (b) (c)

5. 60 + 3 / 30 + 5 — 90 + 8 = 98

40 + 9 / 30 + 1 — 70 + 10 = 80

100 + 30 + 6 / 300 + 40 + 4 — 400 + 70 + 10 = 480

(a) (b)

6. 67 = 50 + 17

− 29 = 20 + 9

30 + 8 = 38

86 = 70 + 16

− 48 = 40 + 8

30 + 8 = 38

7. (17 + 3) + 12 = 32

16 + (12 + 18) = 46

8. (25 + 15) + 10 = 50

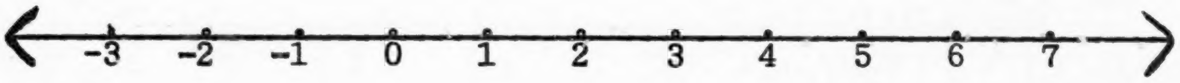
10 + (19 + 21) = 50

2. Adding and Subtracting: Number Line

Objective. To provide pupils with a graphic device to illustrate the meaning of addition and subtraction and to show the relationship between the two operations.

Background Information for the Teacher. In Chapter I, pages 4-15, pupils were introduced to the number line for the purpose of illustrating the orderly arrangement of our numeration system. In this chapter, emphasis will be on the use of the number line to show the operations.

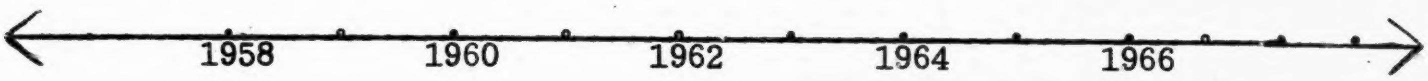
Since there is no limit to the numbers that may be expressed, the number line should be thought of as extending indefinitely in both directions (see illustration).



For our purposes, we will be concerned only with the whole numbers, that is, those numbers beginning with 0 and extending to the right, indefinitely. The numbers to the left of 0 are indicated to illustrate that the number line also extends to the left indefinitely.

When choosing a unit of length, use only that unit to mark off points to the right of 0. You may think of this unit of length as an inch, a half-inch, or any arbitrary length, provided the measure is not "too short" or "too long."

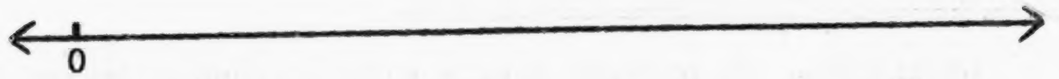
In using the number line, one need not always begin with 0. For example, if we are interested in illustrating the years between 1958 and 1964, only that portion which is significant need be represented (see illustration).



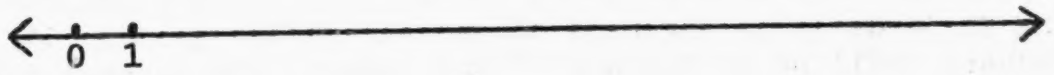
Instruments such as the clock, thermometer, water and gas meters, rulers, protractors, etc. are only few of the many items which use the number line principle. Thus, to look upon the number line as insignificant, contradicts the value of the items mentioned above.

TEACHING PROCEDURE

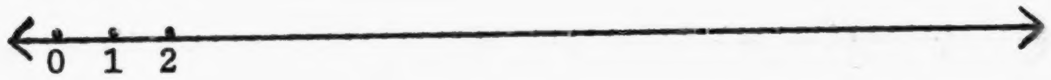
Exploration. On the chalkboard, draw a picture of a number line or use a number line obtained through a commercial source. Choose a point on the number line to represent the number "0." Refer to this point as "zero" or "point zero."



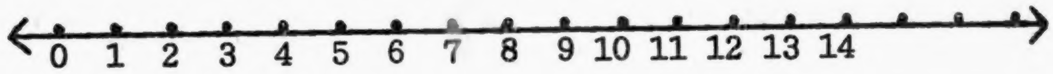
Next, choose a point a short distance to the right of point zero to represent the number 1. Refer to this point as "one" or "point one."



The same distance to the right of point one, mark another point and call it "two" or "point two."



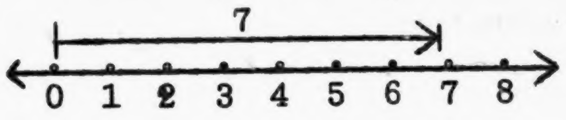
By continuing this process, construct the following:


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

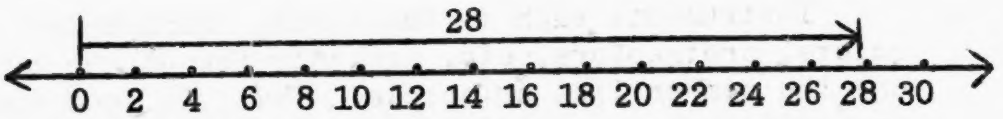
Point out to the pupils that we can show addition (movement to the right) and subtraction (movement to the left) on the number line.

For example:

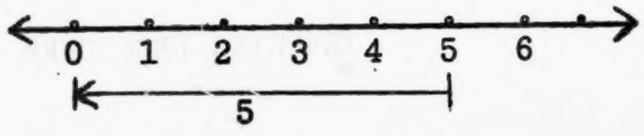
(a) 7 units of length:


7
0 1 2 3 4 5 6 7 8

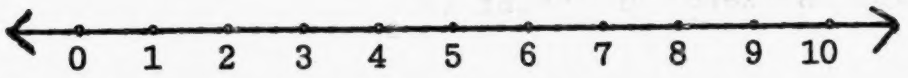
(b) 28 units of length:


28
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30

(c) 5 units of length:



On the chalkboard, represent the following number line:


0 1 2 3 4 5 6 7 8 9 10

Ask the following questions:

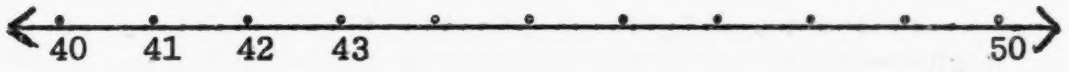
(1) As we look at the number line above, what is the smallest whole number shown? 0

(2) What do we notice as we move to the right along the number line?
The numbers increase. The numbers shown are larger.

(3) Can we tell what whole number will follow the last one shown?
Yes; 11.

(4) What do the arrows on the number line indicate? We have only a part of the line represented; a line continues in both directions without end.

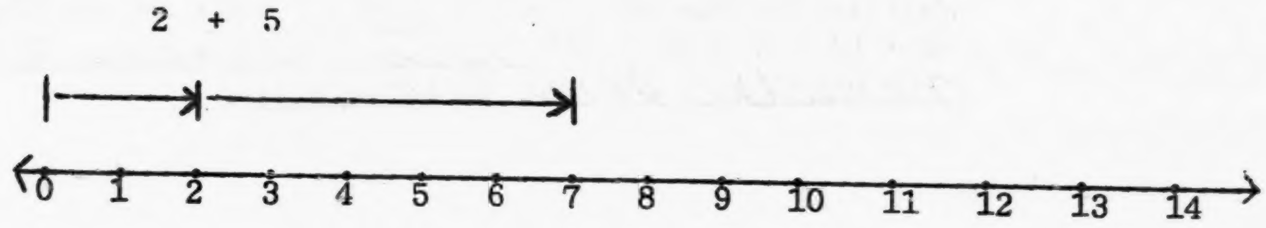
Here is part of a number line showing the whole numbers from 40 through 50.



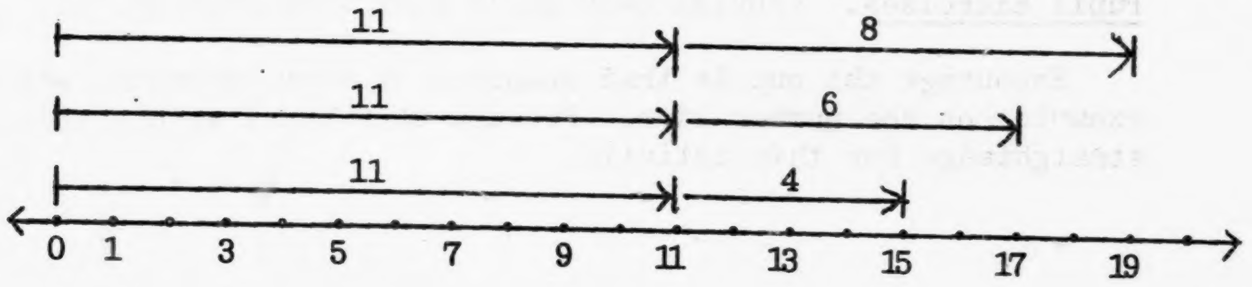
(5) "If we start at point 47 and move to point 42, in which direction do we move?" Left.

(6) "How does the number line show the difference in length from 50 to 46?" 50 is 4 larger than 46 because it is 4 spaces or units to the right of 46.

Next, illustrate to the pupils that the operation of addition can be shown using the number sentence example such as:



Now, point out that adding two-place and one-place numbers is possible, for example:



Ask several pupils to come to the chalkboard and write in the missing numeral for these number sentences and compare them with the above illustration.

(a) 11 + 4 = 15

(b) 11 + 6 = 17

(c) 11 + 8 = 19

Have other pupils, one at a time, come to the chalkboard and illustrate on the number line the following examples:

(a)

22 + 5 = 27

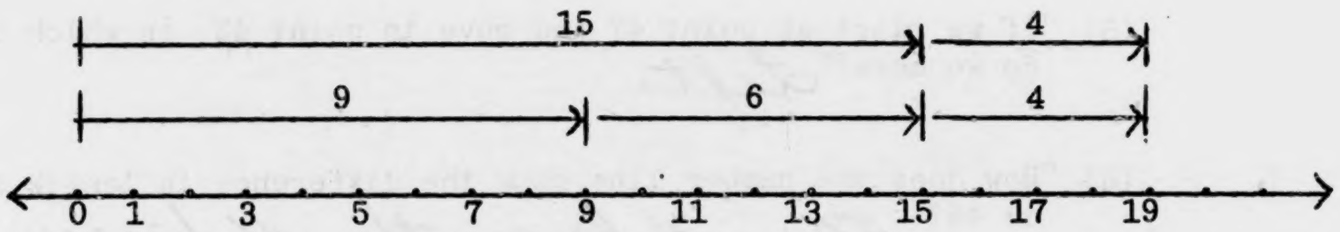
13 + 8 = 21

(b)

19 + 7 = 26

21 + 8 = 29

Following periods of practice with addition, introduce the pupils to higher-decade addition in adding three numbers. Construct the following on the chalkboard:



Have the pupils study the above illustration so that they may answer the following questions:

"Look at the number line. What is the sum of 9 + 6? 15 Are 15 and 9 + 6 two names for the same number? yes Then, what is the sum of 15 + 4? 19 Can we now say that (9 + 6) + 4 = 15 + 4?" yes, because they both name the same number.

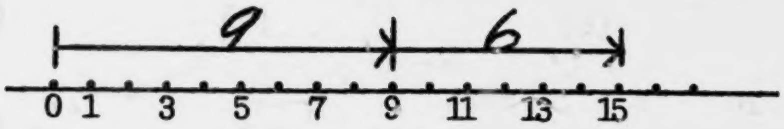
Pupil Exercises. Provide each pupil with worksheet No. 11.

Encourage the pupils that accuracy is very important when working examples on the number line. Provide each pupil with a ruler or a straightedge for this activity.

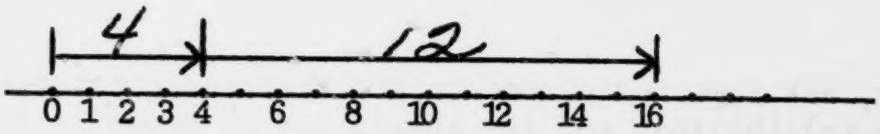
Worksheet No. 11

WORK EACH EXAMPLE ON THE NUMBER LINE, THEN WRITE THE MISSING NUMERAL TO COMPLETE THE SENTENCE.

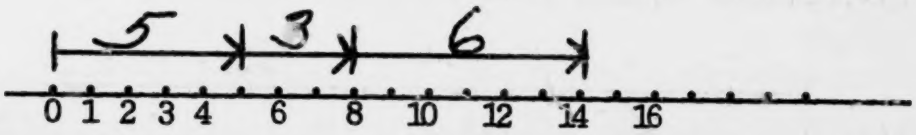
1. 9 + 6 = 15


9
6
0
1
3
5
7
9
11
13
15

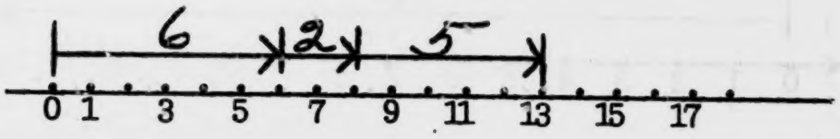
2. 4 + 12 = 16


4
12
0
1
2
3
4
6
8
10
12
14
16

3. (5 + 3) + 6 = 14


5
3
6
0
1
2
3
4
6
8
10
12
14
16

4. 6 + (2 + 5) = 13

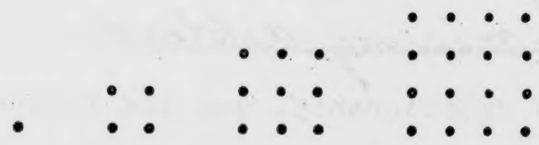

6
2
5
0
1
3
5
7
9
11
13
15
17

WRITE THE MISSING NUMERAL FOR EACH EXAMPLE. (Construct and use the number line if necessary.)

	(a)	(b)
5.	23 + 6 = 29	(14 + 5) + 6 = 25
6.	43 + 8 = 51	(6 + 7) + 9 = 22
7.	27 + 35 = 62	22 + (19 + 6) = 47
8.	38 + $\boxed{22}$ = 60	($\boxed{6}$ + 9) + 20 = 35
9.	$\boxed{29}$ + 16 = 45	23 + 27 = 50
10.	17 + 83 = 100	65 + $\boxed{45}$ = 110

DISCOVER THE PATTERN:

11.



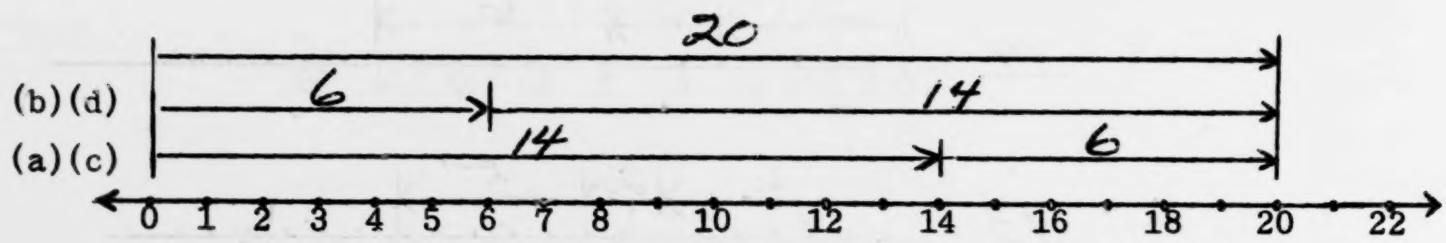
1 4 9 16 25 36 49 64

To show the operation of subtraction on the number line and the inverse relationship between addition and subtraction, use the following examples for analysis:

Addition

(a)	(b)	(c)	(d)
$14 + 6 = 20$	$6 + 14 = 20$	$\begin{array}{r}14\\ +\ 6\\ \hline 20\end{array}$	$\begin{array}{r}6\\ +\ 14\\ \hline 20\end{array}$

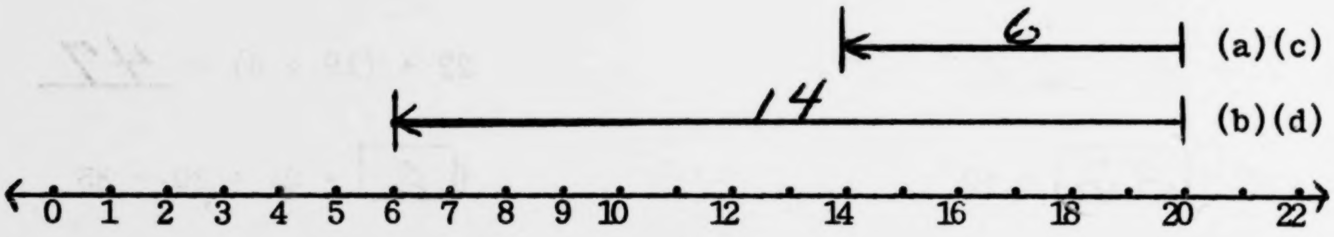
Now, ask someone to come to the chalkboard to show how the addition examples (a),(b),(c), and (d) above, can be shown on the number line. Illustrations should look similar to this:


20
(b)(d)
6
14
(a)(c)
14
6
0 1 2 3 4 5 6 7 8 10 12 14 16 18 20 22

Now, construct a second number line and use the following examples to show the inverse relationship between addition and subtraction:

Subtraction

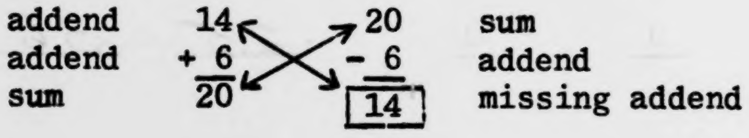
(a)	(b)	(c)	(d)
$20 - 6 = 14$	$20 - 14 = 6$	$\begin{array}{r}20\\ -\ 6\\ \hline 14\end{array}$	$\begin{array}{r}20\\ -\ 14\\ \hline 6\end{array}$


6
(a)(c)
14
(b)(d)
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20 22

Now ask the pupils, "In which direction must we move to show subtraction on the numer line? To the left. Why?"

In addition we move to the right and get a large number we call the sum. In subtraction we move to the left beginning with the sum and finding the missing number or missing addend.

To show another example of the inverse relationship, use the following illustration:


addend
addend
sum
14
+ 6
20
20
- 6
14
sum
addend
missing addend

This may also be written in the form:

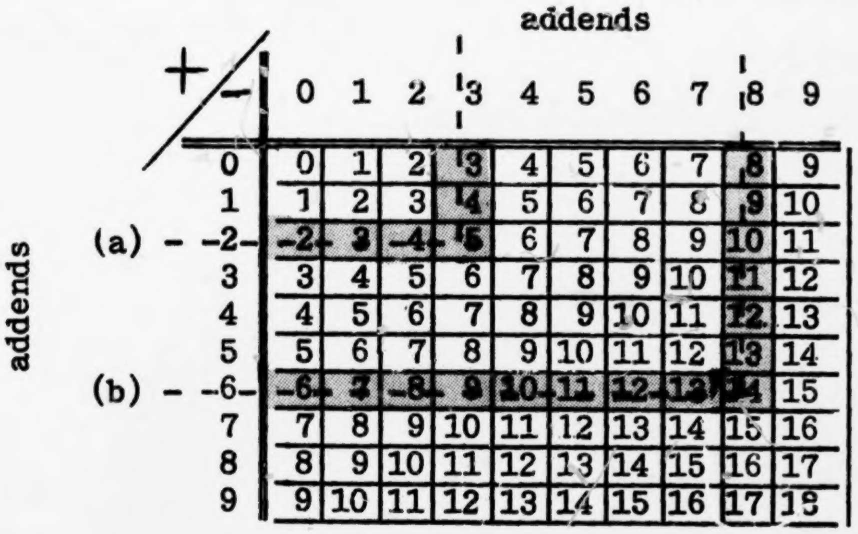
14 + 6 = 20 　　　　 20 - 6 = $\boxed{14}$

After examining several subtraction examples on the chalkboard with the pupils, have them practice the following at their desks, using the number line. (Write these on the chalkboard.)*

(a)	(b)
14 + 9 = $\boxed{23}$	32 - 24 = $\boxed{8}$
23 - 14 = $\boxed{9}$	19 - $\boxed{9}$ = 10
33 - $\boxed{11}$ = 22	$\boxed{19}$ - 15 = 4

The pupils will probably show their work in a variety of ways for each example shown above. Discuss the work thoroughly. Let each pupil with a different illustration explain his work.

Another illustration for showing the relationship between addition and subtraction is through the use of the Addition-Subtraction Chart. (See illustration.)


addends
addends
(a)
(b)

+ / −	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
(a) 2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
(b) 6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

The numerals at the top and on the left of the chart are the addends.

---

*The discovery of more difficult subtraction combinations on the number line certainly does not mean that the pupils will retain them after seeing the illustration once. Thus, practice will be necessary, although the type and amount of practice needed will differ for individual members of the group. The use of the number line will enable students to "reason" out an answer when the number combination is unknown or forgotten.

In the illustration, example (a) can be interpreted as follows:

Addition	Subtraction
2 + 3 = 5	5 - 3 = 2
3 + 2 = 5	5 - 2 = 3

In example (b) we have:

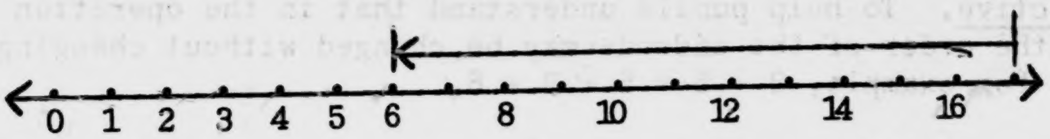
6 + 8 = 14	14 - 8 = 6
8 + 6 = 14	14 - 6 = 8

The point of intersection for addition, example (a), is 5 and the point of intersection for example (b), is 14. To subtract, we begin at the point of intersection, subtracting one addend and finding the missing addend.

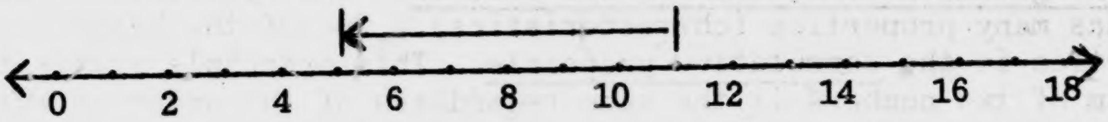
Pupil Exercises. Provide each pupil with worksheet No. 12.

Make sure that each pupil has a ruler or a straightedge for this activity. With the less able group, it may be necessary for the teacher to read problems 6, 7, and 8 with the pupils.

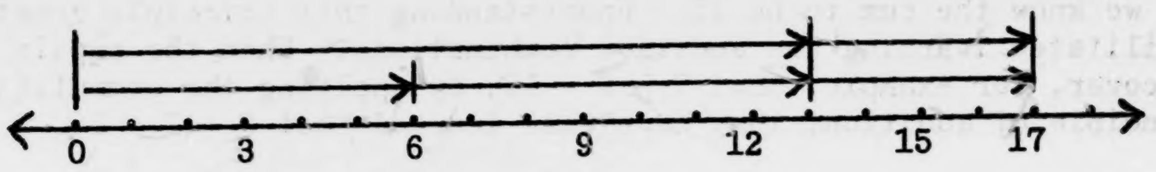
WRITE THE NUMBER SENTENCE FOR EACH ILLUSTRATION WHICH IS SHOWN ON THE NUMBER LINE. USE THE BLANK SPACE AT THE RIGHT OF EACH ILLUSTRATION.

1. 

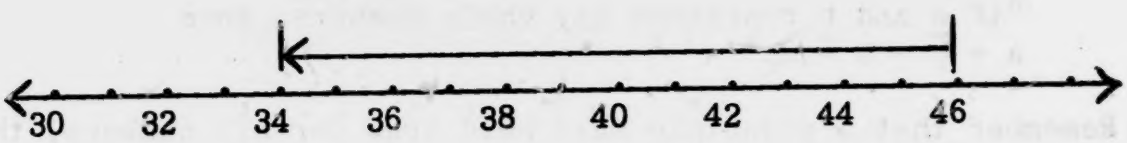
17-11=6

2. 

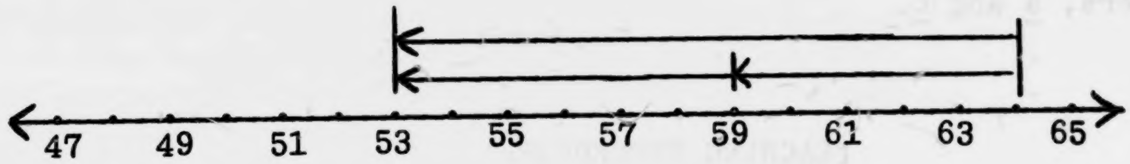
11-6=5

3. 

(6+7)+4=17

4. 

46-12=34

5. 

(64-5)-6=53

SOLVE EACH PROBLEM. USE THE NUMBER SENTENCE FORM.*

6. The game's leaders and the office monitors from Montezuma School were rewarded for their work with a free trip to the San Diego Zoo. Twenty-two boys and 18 girls made the trip. How many children made the trip?

22 + 18 = 40

7. At the zoo, a toy salesman had 86 toys to sell. By late afternoon, he had only 13 left. How many toys had he sold?

86-☐=13; ☐=73

8. The children traveled to the zoo by bus, a distance of 7 miles. At the zoo they traveled another 10 miles. Before returning to school, they took a 27-mile sight-seeing tour. What was the total distance traveled that day?

(7+10)+27=44

---

* Number sentences will vary.

3. Commutative Principle of Addition

Objective. To help pupils understand that in the operation of addition the order of the addends may be changed without changing the sum. For example, 3 + 5 = 5 + 3 = 8.

Background Information for the Teacher. The operation of addition has many properties (characteristics). One of the basic properties is the commutative principle. This principle states that the sum of two numbers is the same regardless of the order in which the numbers are added. This means that 12 + 3 = 3 + 12 whether or not we know the sum to be 15. Understanding this principle greatly facilitates learning the addition combinations. When the pupils discover, for example, that 7 + 9 = 16, by applying the commutative principle of addition, they have also learned that 9 + 7 = 16.

This principle may also be stated as:

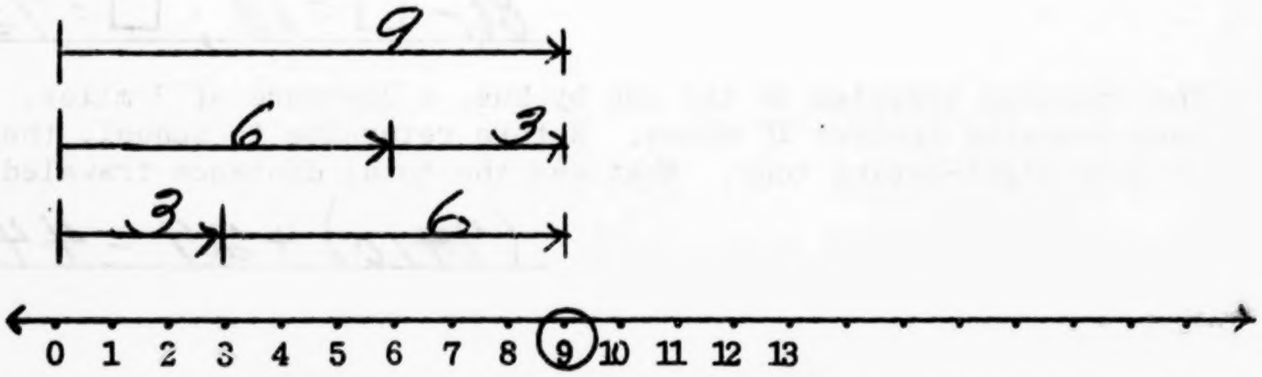
> "if a and b represent any whole numbers, then
> a + b = b + a."

Remember that a principle must hold true for all numbers, that is, the statement a + b = b + a is true for all pairs of whole numbers, a and b.

TEACHING PROCEDURE

Exploration. Non-numerical and numerical examples of combining things should be introduced by using concrete objects such as erasers and rulers to indicate that two sets (groups) of objects may be put together in two different ways. Emphasize that the order of putting sets of objects together does not affect the number of objects.

Next, use the number line to illustrate the following number sentence example: 3 + 6 = 6 + 3 = 9



Point out to the pupils that in using the number line we can reach agreement that the number line illustration is useful in thinking of both 3 + 6 = 9 and 6 + 3 = 9. If this is established early in the use of this unit, pupils will have a graphic representation of the commutative principle of addition. (The use of the phrase commutative principle with pupils of this grade level is optional.)

Use other examples to illustrate the commutative principle of addition, that is,

(a) 14 + 12 = 12 + 14

(b) 24 + 10 = 10 + 24

(c) 36 + 14 = 14 + 36.

Next, indicate to the pupils that the order of addends in arithmetic as compared to the order of letters in the English language and the order of events or ways of doing things sometimes vary. For example, ask the following questions:

1. Pretend you have a dime and a quarter. You are paying for a 35¢ book with a quarter and a dime. Would the order of giving the coins to the clerk make a difference in what you paid him? No

2. Use the letters "o" and "n" to make two words. What are the two words? on and no. Did the order of the letters change the result? yes.

3. Is the order important when you:

   (a) Season your food with salt and pepper? No
   (b) Put your socks and shoes on? yes
   (c) Roast weiners and eat them? yes
   (d) Fix a flat and ride your bicycle? yes

4. What can we say about the order of addends in arithmetic as compared to the order of letters in the English language and the order of events or ways of doing things? In arithmetic changing the order of addends does not affect the sum; in the English language and in the order of events, the results are not always the same.

5. What can we say about each of the following? (Write these on the chalkboard.)

   (a) 40 + 50 = 50 + 40
   (b) 69 + 8 = 8 + 69
   (c) 687 + 596 = 596 + 687
   (d) 1,000,000 + 1 = 1 + 1,000, 000

The order of addends is changed; however, the sum in each case is not affected. Each is a true statement.

The important outcome of these examples is to show that the order of adding two whole numbers does not change the sum.

Encourage pupils to state this principle in their own words. Once they have mastered the meaning of this generalization and can make a statement, inform them that the name for this principle is the commutative principle of addition.

To discover if the commutative principle of addition applies to subtraction, display 12 objects and ask the pupils to "take away" 7 objects. Also, ask why it is possible to get a difference of 5. Next, direct their attention to changing the idea; that is, to see if 12 objects can be "taken away" from 7 objects.

On the chalkboard, write the expression $12 - 7 = 7 - 12$ and ask: "Will someone explain why this is not a true statement? Can someone show us this idea through the use of objects?"

Next, encourage the pupils to examine the following illustrations to see if the commutative principle always holds true as it does in addition. Consider these subtraction examples:

(a)	(b)
$12 - 7 = 5$	$63 - 13 = 50$
$7 - 12 = \square$	$13 - 63 = \square$

Ask the following questions:

(1) In illustration (a), $12 - 7 = 5$, if we change the order of the numbers to be subtracted so that we have $7 - 12 = \square$, do we get a whole number for an answer? No

(2) Are the answers to $12 - 7 = 5$ and $7 - 12 = \square$ the same? No

(3) Examine illustration (b). If we change the order of the numbers to be subtracted, do we get a whole number for an answer? No

(4) Does the commutative principle of addition always hold true? Yes

(5) Is there a commutative principle for subtraction? Defend your answer. *

---

*For those pupils who now immediately want to assume that there exists a principle for subtraction based on the above, ask, "Does verification for one or even several examples establish a principle?" A pupil needs to recognize that a principle is a generalization about numbers that must hold true in every instance. Since we can find instances where the generalization is false, showing it is not true in each instance, there is no commutative principle for subtraction. (Use this same explanation when discussing the operation of division, also.)

Next, review with the pupils the Addition-Subtraction Chart on page 67 which was introduced earlier. When the pupils become familiar with this chart in locating patterns, introduce them to "magic grids." Shown below is a grid with 15, 6, 8, and 18. Encourage pupils to find the pattern by adding two numbers whose numerals appear in the interior region of the grid. Have them write each sum in two places, thus showing that the sum is the same regardless of the order in which the numbers are added. Emphasize that in the "magic grid" we may add up and down, across in either direction, and on the diagonal.

33	23	24	14
21	15	6	21
26	8	18	26
14	23	24	33

This grid shows that numbers can be paired and added regardless of order. For example, 15 + 6 = 6 + 15; 8 + 18 = 18 + 8; 15 + 18 = 18 + 15; etc.

Have the pupils study the magic grid and ask, "How many addition combinations are shown? 12 Is a subtraction sentence related to each addition sentence?" yes; for example, if 6 + 18 = 24, then 24 - 18 = 6 and 24 - 6 = 18.

On the chalkboard construct a magic grid with only the numeral 6 shown in the interior of the grid and the numerals 15, 14, and 13 shown in the exterior.

16	15	15	14
13	7	6	13
17	8	9	17
14	15	15	16

Ask, "Can we find what the other interior numerals are if the exterior numerals represent the sum?" yes; for example, 15 - 6 = 9; 14 - 6 = 8; 13 - 6 = 7; etc.

Encourage pupils to complete the "magic grid" by adding any two numbers whose numerals appear in the interior region of the grid. Have them write each sum in two places in the exterior region.

Construct the following grids on the chalkboard and have the pupils put in the missing numerals:

11	44	23	56
22	5	17	22
45	39	6	45
56	44	23	11

16	21	32	37
30	7	23	30
23	14	9	23
37	21	32	16

Pupil Exercises. Provide each pupil with worksheet No. 13.

Limit the explanation of this worksheet with the more able pupils. Ask them to interpret the directions. If their interpretation is correct, let them proceed independently. With the less able group, offer as much explanation as is necessary. Make example 9 optional for the less able group.

USE THE COMMUTATIVE PRINCIPLE OF ADDITION TO COMPLETE EACH NUMBER SENTENCE.

	If	Then
1. 12 + 5 = 5 + 12	12 + 5 = 17	5 + 12 = 17
2. 43 + 26 = 26 + 43	43 + 26 = 69	26 + 43 = 69
3. 67 + 12 = 12 + 67	67 + 12 = 79	12 + 67 = 79
4. 42 + 18 = 18 + 42	42 + 18 = 60	18 + 42 = 60
5. 45 + 23 = 23 + 45	45 + 23 = 68	23 + 45 = 68

DRAW A CIRCLE AROUND THE LETTER OF EACH EXAMPLE WHICH SHOWS THE COMMUTATIVE PRINCIPLE OF ADDITION.

6. [✓ a.] 76 + 13 = 13 + 76
   b. 84 + 29 = 84 + 29
   [✓ c.] 19 + 37 = 37 + 19
   [✓ d.] 5 + (10 + 4) = (10 + 4) + 5
   [✓ e.] (4 + 5) + 10 = 10 + (4 + 5)
   f. 27 + 43 = 34 + 72
   [✓ g.] 0 + 18 = 18 + 0
   h. 1 + 0 = 1 + 0

WRITE TWO NUMBER SENTENCES FOR EACH OF THE FOLLOWING:

7. If 45 pupils from McDowell School went by bus to Disneyland and 23 pupils went in several family cars, how many pupils went altogether?

45 + 23 = 68 or 23 + 45 = 68

8. Twenty-three pupils went to Disneyland in 5 different cars. Forty-five pupils went by bus. How many pupils went to Disneyland altogether?

23 + 45 = 68 or 45 + 23 = 68

WRITE A "word problem" FOR THE FOLLOWING NUMBER SENTENCE: 65 + 17 = 82*

9. ____

---

*Problems will vary.

4. Associative Principle of Addition

Objectives. (1) To help pupils understand that when adding numbers, various groupings are permissible. (2) To help pupils understand that addition is a binary operation; namely, just two numbers can be added in one addition operation.

Background Information for the Teacher. When adding three or more numbers, the addends may be grouped or paired together in more than one way without affecting the sum. For example, in adding 2 + 3 + 4, we may group the addends as (2 + 3) + 4 or as 2 + (3 + 4) and in either case will get the sum of 9. In the first case, we may say 5 + 4 = 9, and in the second, 2 + 7 = 9. This principle is named the associative principle of addition.

Since addition is a binary operation, an equation such as $2 + 3 + 4 = \square$ does not tell us what numbers to group or which addition operation to perform first. However, because of the associative principle of addition, we may group the addends as $(2 + 3) + 4 = \square$ and $2 + (3 + 4) = \square$ without affecting the sum. In both cases, we are really dealing with two numbers at a time.

Parentheses are used to indicate which operation is to be performed first. We shall use it for the sole purpose of indicating (a) what numbers are grouped together and (b) what operation is to be performed first.

When adding two numbers whose sum is greater than 9, it is often convenient to rename the numbers. For instance, notice how different names for the numbers five and ten and how the associative principle of addition are used in the following example:

$9 + 5 = \square$	
$9 + (1 + 4) = \square$	Rename 5 as (1 + 4)
$(9 + 1) + 4 = \square$	Associate addends
$10 + 4 = \square$	Adding 9 + 1
$14 = \square$	Place-value (10 + 4 means 14)

It is possible to always have the numeral or multiple of 10 appear first since the order of the addends does not affect the sum. For example, the addition of $5 + 9 = \square$ can be done in the following ways:

(a)	(b)
$5 + 9 = \square$	$5 + 9 = \square$
$5 + (5 + 4) = \square$	$9 + 5 = \square$
$(5 + 5) + 4 = \square$	$9 + (1 + 4) = \square$
$10 + 4 = \boxed{14}$	$(9 + 1) + 4 = \square$
	$10 + 4 = \boxed{14}$

TEACHING PROCEDURE

Exploration. On the chalkboard write the number sentence 6 + 7 + 5 = ☐. Let the pupils experience for themselves that no matter which grouping they choose, the first step to the solution of the number sentence may be the addition of two of the three numbers.

Since the pupils have been introduced to parentheses earlier in the unit, it might be helpful to review the use of and the need for parentheses.

Next, ask, "Is it possible to add three or more numbers at a time?" No, because addition is a process of thinking about two numbers and getting a third number.

Using parentheses, "Who can show us several examples of grouping the numbers in this number sentence which is written on the chalkboard? Do not change the arrangement of the numerals." Responses should be:

(6+7)+5 = ☐
6+(7+5) = ☐

Now have the pupils examine the following examples which you have written on the chalkboard:

(a) 15 + 5 + 12 = ☐

(b) 33 + 17 + 19 = ☐

(c) 14 + 18 + 12 = ☐

Ask the following questions:

1. "In example (a), what two numbers should be grouped to make addition easier? (15+5) + 12 = ☐

   Would the sum change if we grouped 15 + (5 + 12) = ☐ ?" No

2. "What numbers should we group in example (b)? (33+17)+ 19 = ☐
   If we group 33 + (17 + 19) = ☐, what would happen to the sum?"

   It would be the same.

3. "What grouping would be best in example (c)? 14+(18+12) = ☐
   Is it true that (14 + 18) + 12 is equal to 14 + (18 + 12)?"

   Yes, because the sum is the same.
   Yes, because of the associative principle of addition.

Point out to the pupils that in each example the arrangement of the numerals was not changed; only parentheses were used to group two numbers.

For practice, write several examples on the chalkboard and have the pupils copy and solve each example. Have them use parentheses and show two different groupings. In grouping each example in two ways, have them observe what happens to the sum. Indicate that the arrangement of the numerals must not be changed. The teacher may want to use the following examples:

(a) $14 + 16 + 22 = \square$ (c) $16 + 19 + 1 = \square$

(b) $37 + 13 + 10 = \square$ (d) $23 + 22 + 8 = \square$

A sample of each pupil's work should look similar to the following:

$(14 + 16) + 22 = \square$ $14 + (16 + 22) = \square$

$30 + 22 = \underline{52}$ $14 + 38 = \underline{52}$

When examples similar to the above have been completed, the teacher may ask leading questions which might guide the pupils to make a generalization. For example, ask:

(1) "Why are parentheses useful in arithmetic?"

(a) They help to group two numbers.
(b) They indicate which operation (which group) we must work with first.

(2) "Was the arrangement of numerals changed?" No.

(3) "What can we say about the order of grouping numbers in addition?"

The order of grouping does not affect the sum

Inform the pupils that they have been working with a principle which is called the associative principle of addition.

If some pupils have difficulty understanding this principle, the teacher may find the number line helpful in graphically demonstrating that "numbers to be added may be grouped in different ways without affecting the sum."

Pupil Exercises. Provide each pupil with worksheet No. 14.

Limit the explanation of this worksheet with the more able pupils. Ask them to interpret the directions. If their interpretation is correct, let them proceed independently. With the less able group, offer as much explanation as is necessary.

Interim Mathematics Unit - Grade 4
San Diego City Schools (1-65)

Worksheet No. 14

USE PARENTHESES TO SHOW THAT THE ORDER IN WHICH NUMBERS ARE GROUPED DOES NOT AFFECT THE SUM. FOR EXAMPLE:

$$(12 + 28) + 10 = 12 + (28 + 10)$$
$$40 + 10 = 12 + 38$$
$$50 = 50$$

	(a)	(b)
1.	$(14 + 9) + 22 = 14 + (9 + 22)$ $23 + 22 = 14 + 31$ $45 = 45$	$(25 + 17) + 13 = 25 + (17 + 13)$ $42 + 13 = 25 + 30$ $55 = 55$
2.	$(75 + 5) + 20 = 75 + (5 + 20)$ $80 + 20 = 75 + 25$ $100 = 100$	$(63 + 17) + 13 = 63 + (17 + 13)$ $80 + 13 = 63 + 30$ $93 = 93$

WRITE THE MISSING NUMERAL FOR EACH ○ WHICH WILL MAKE THE NUMBER SENTENCE TRUE.

	(a)	(b)
3.	$12 + 9 = 9 + 12$	$23 + 35 = 35 + 23$
4.	$65 + 21 = 21 + 65$	$8 + (7 + 6) = (8 + 7) + 6$
5.	$(18 + 7) + 6 = 18 + (7 + 6)$	$(33 + 19) + 18 = 33 + (19 + 18)$

SOLVE THESE EXAMPLES. (Hint: Think of the addition principles which you have learned.)

	(a)	(b)
6.	$33 + 59 = 33 + 50 + \boxed{9}$	$57 + 25 = 57 + \boxed{20} + 5$
7.	$38 + 66 = 30 + \boxed{8} + 66$	$67 + 55 = 67 + 40 + \boxed{15}$
8.	$23 + \boxed{0} + 57 = 23 + 57$	$\boxed{5} + 84 + 8 = 89 + 8$

DRAW A CIRCLE AROUND THE LETTER OF EACH EXAMPLE WHICH IS TRUE.

9. ⓐ (5 + 3) < 10 ⓒ (17 + 17) < 37 e. (1 + 0) < (0 + 1)
ⓑ (15 + 8) < 25 ⓓ (99 + 0) < 100 ⓕ 23 < (17 + 7)

5. Extending Addition and Subtraction

Objectives. (1) To extend the regrouping concept to include 10 tens as a hundred and 10 hundreds as a thousand. (2) To promote a better understanding of the conventional algorithms of addition and subtraction.

Background Information for the Teacher. It is important for pupils to understand that the idea of "different names for the same number" is basic for understanding and learning the operations of addition and subtraction.

Consider the following example which should make clear (1) the need to learn how to express numbers in different forms and (2) the useful application of this skill in addition and subtraction:

> "Find the number that must be added to 376 so that 523 will be the sum." If the sentence is written as $376 + \boxed{} = 523$, immediate recall of the subtraction combinations is unlikely. Hence, it is helpful to express 523 and 376 in expanded form, that is,

$$\begin{array}{rcccccc}
523 & = & 500 + 20 + 3 & = & 500 + 10 + 13 & = & 400 + 110 + 13 \\
-\;\underline{376} & = & \underline{300 + 70 + 6} & = & \underline{300 + 70 + 6} & = & \underline{-(300 + 70 + 6)} \\
\boxed{} & & & & & = & 100 + 40 + 7 \\
 & & & & & = & 147
\end{array}$$

The primary issue is that of recognizing the need for more ones in the ones' place and more tens in the tens' place.

In the following example there is less difficulty. We simply name the addends to exhibit the number of ones, the number of tens, etc. Then the ones, tens, hundreds, etc. are added. For example:

$$\begin{array}{rcccl}
249 & = & 200 + 40 + 9 & & \\
+\;676 & = & \underline{600 + 70 + 6} & & \\
 & = & 800 + 110 + 15 & = & (800 + 110) + 15 \\
 & & & = & 910 + 15 \\
 & & & = & 925
\end{array}$$

Here the numeral (800 + 110 + 15) is expressed as the numeral by applying properties of our system of numeration and the commutative principle of addition and the associative principle of addition.

TEACHING PROCEDURE

Exploration. On the chalkboard write the following problem:

> "Cindy put 89 animal pictures in her social studies album on Saturday and 71 pictures on Sunday. How many pictures did she mount in the two days?"

Next, write the problem in expanded form for analysis such as:

89 = ☐ tens, ☐ ones
+ 71 = ☐ tens, ☐ ones
___ tens, ___ ones

Ask the following questions: *

1. How many ones do we get when we add? 10
2. Can we regroup 10 ones as 1 ten, 0 ones? Yes
3. If we add 1 ten to 8 tens + 7 tens, how many tens will we have? 16
4. Is it true that 89 + 71 is 16 tens, 0 ones? Yes
5. How many hundreds, tens, and ones in 16 tens and 0 ones? 1 hundred, 6 tens, 0 ones

To show regrouping of 10 tens as a hundred, write the following example on the chalkboard and guide the pupils in analyzing the process through questions such as: *

264 = ___ hundreds, ___ tens, ___ ones
+182 = ___ hundred, ___ tens, ___ ones
= ___ hundreds, ___ tens, ___ ones

1. Do we need to regroup the ones? No
2. How can we regroup the sum of 6 tens + 8 tens? 1 hundred + 4 tens
3. When we add the 1 hundred to the other hundreds, how many hundreds, tens, and ones will we have?" 4 hundreds, 4 tens, 6 ones
4. What is the sum of 264 + 182 = ☐ ? 446

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* As the pupils respond to each question, write the numeral in each appropriate frame.

Using the same pattern of analysis, the teacher may wish to use the following example to illustrate regrouping in the ones', tens', and hundreds' place:

818 = 8 hundreds, 1 ten, 8 ones
+ 383 = 3 hundreds, 8 tens, 3 ones
= 11 hundreds, 9 tens, 11 ones
= 11 hundreds, 10 tens, 1 one
= 12 hundreds, 0 tens, 1 one
= 1 thousand, 2 hundreds, 0 tens, 1 one
= 1000 + 200 + 0 + 1
= 1,201

The example shown above presents a detailed analysis of regrouping. Time should be taken to work the example cooperatively with the pupils. When the concept is well understood, write several examples on the chalkboard for the pupils to do at their desks. Examples such as these will be appropriate:

$$\begin{array}{r} 895 \\ +\ 567 \\ \hline \end{array} \qquad \begin{array}{r} 677 \\ +\ 323 \\ \hline \end{array} \qquad \begin{array}{r} 723 \\ +\ 489 \\ \hline \end{array}$$

Regrouping tens and hundreds to subtract is an extension of the developmental work initiated earlier in this chapter. You may wish to introduce this regrouping by writing the following problem on the chalkboard:

> "Bob lives near a lake where men from the Wildlife Service catch wild ducks. The men attach bands to the ducks in order to study their living habits. One day 328 ducks were banded. The next day only 134 ducks were banded. How many more ducks were banded on the first day than on the second day?"

Through carefully planned, guided discussion questions, have the pupils assist in stating the problem, using the long form, that is, *

328 = ____ hundreds, ____ tens, ____ ones
- 134 = ____ hundred, ____ tens, ____ ones
____ hundreds, ____ tens, ____ ones

Ask, "How can we regroup 328 so that we can subtract the tens? 2 hundreds 12 tens, 8 ones. Did regrouping take place in the ones' place? No If we write the problem in number sentence form such as 328 - 134 = ☐, what is the missing numeral (difference)? 194

To illustrate regrouping both the tens and the hundreds, use the following example: *

932 = ____ hundreds, ____ tens, ____ ones
- 576 = ____ hundreds, ____ tens, ____ ones
____ = ____ hundreds, ____ tens, ____ ones

Ask, "How can we regroup 932 before we can subtract?" First, change 932 to 9 hundreds, 2 tens, 12 ones; this will make subtraction in the ones' place possible. Next, change the 9 hundreds, 2 tens to 8 hundreds, 12 tens; this makes it possible to complete the subtraction process.

After completing the subtraction, as, "What numeral is another name for 3 hundreds, 5 tens, 6 ones?" 356

For individual practice, have pupils work several of the following examples at their desks:

(a)	(b)	(c)	(d)
819	829	589	904
- 453	- 547	- 390	- 576

Pupil Exercises. Provide each pupil with worksheet No. 15.

Encourage pupils to see relationships with respect to the operation of addition and subtraction. The less able group may need more introductory work before they can do these examples independently.

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* As the pupils respond to your questions, write the numerals in the appropriate blank spaces.

FIND THE MISSING NUMERAL FOR EACH BLANK SPACE IN THE FOLLOWING EXAMPLES:

	Addition	Subtraction
1.	$$\begin{array}{rl} 68 = & 6 \text{ tens, } 8 \text{ ones} \\ +\,87 = & 8 \text{ tens, } 7 \text{ ones} \\ \hline & 14 \text{ tens, } 15 \text{ ones} = 155 \end{array}$$	$$\begin{array}{rl} 155 = & 14 \text{ tens, } 15 \text{ ones} \\ -\,87 = & 8 \text{ tens, } 7 \text{ ones} \\ \hline & 6 \text{ tens, } 8 \text{ ones} = 68 \end{array}$$
2.	$$\begin{array}{rl} 143 = & 100 + 40 + 3 \\ +\,58 = & \quad\;\; 50 + 8 \\ \hline & 100 + 90 + 11 = 201 \end{array}$$	$$\begin{array}{rl} 201 = & 100 + 90 + 11 \\ -\,58 = & \quad\;\; 50 + 8 \\ \hline & 100 + 40 + 3 = 143 \end{array}$$

3. What discovery did you make about examples 1 and 2 above? Addition and subtraction are related. In addition we must find the sum; in subtraction we want to find the missing addend.

DO THE FOLLOWING EXAMPLES:

	(a)	(b)
4.	$25 + \boxed{41} = 66$	$66 - 25 = \boxed{41}$
5.	$\boxed{70} + 16 = 86$	$86 - 70 = \boxed{16}$
6.	$114 + 38 = \boxed{152}$	$152 - 114 = \boxed{38}$
7.	$50 + \boxed{75} = 125$	$125 - 75 = \boxed{50}$
8.	$163 + \boxed{37} = 200$	$200 - 163 = \boxed{37}$

9.

(a)	(b)	(c)	(d)
$\begin{array}{r} 5\,8\,4 \\ -\,2\,3\,\boxed{2} \\ \hline \boxed{3}\,\boxed{5}\,2 \end{array}$	$\begin{array}{r} 3\,5\,2 \\ +\,\boxed{2}\,3\,2 \\ \hline 5\,\boxed{8}\,\boxed{4} \end{array}$	$\begin{array}{r} 5\,2\,7 \\ +\,4\,1\,8 \\ \hline 9\,4\,5 \end{array}$	$\begin{array}{r} \boxed{9}\,\boxed{4}\,\boxed{5} \\ -\,4\,1\,8 \\ \hline 5\,2\,7 \end{array}$

FIND A PATTERN:

10. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024

B. Multiplication and Division

Multiplication of Whole Numbers

Objective. To review and extend the meaning of multiplication, using sets, arrays, and number sentences.

Background Information for the Teacher. Multiplication of whole numbers can be defined in terms of sets. For example, the illustration below represents an array[*] of 3 rows and 4 columns. The number associated with this array is 3 x 4. Since 3 x 4 is not a standard form for expressing a counting number, we can set up a one-to-one correspondence between the members of the set (illustration) and the members of the set of counting numbers. Thus, the number property of the 3 x 4 array is 12. We can then name

	4 members (columns)			
	*	*	*	*
3 sets (rows)	*	*	*	*
	*	*	*	*

the number of the illustration as 3 x 4 or 12.

In 3 x 4 = 12, the 3 and 4 are factors; the 12 is called the product of 3 x 4. Thus we have:

Factor		Factor		Product
3	x	4	=	12

It is often useful to think of multiplication in terms of repeated addition. For example,

3 x 5 is equal to 5 + 5 + 5 or three 5's

6 x 2 is equal to 2 + 2 + 2 + 2 + 2 + 2 or six 2's

The repeated addition approach to multiplication has limitations. Using repeated addition to solve the problem "how many quarts are there in 172 gallons" is too time consuming, since 4 would be used as an addend 172 times. Thus, the operation of multiplication is necessary which provides the element of economy.

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[*] See Glossary, page 134.

TEACHING PROCEDURE

Exploration. Introduce the pupils to the word array and show several illustrations which may represent an array. Empty egg cartons may serve as good representations. Proceed from a physical model to the graphic representation, for example:

Array Examples	Definition	Meaning	Number Sentence
✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻	3 sets with 4 members in each set	three 4's	3 x 4 = 12
✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻ ✻	2 sets with 6 members in each set	two 6's	2 x 6 = 12

Next, represent other arrays on the chalkboard and have pupils respond orally, identifying each example in several ways. Example (a) will serve as a model:

(a)	(b)	(c)
✻ ✻ ✻ ✻ ✻	. . . . . .	△ △ △
✻ ✻ ✻ ✻ ✻	. . . . . .	△ △ △
✻ ✻ ✻ ✻ ✻	. . . . . .	△ △ △
	. . . . . .	△ △ △
3 × 5 = 15	4 x 6 = 24	4 x 3 = 12
three 5's = 15	four six's = 24	four 3's = 12
5 + 5 + 5 = 15	6+6+6+6 = 24	3+3+3+3 = 12
3 + 3 + 3 + 3 + 3 = 15		4 + 4 + 4 = 12

Write the following examples on the chalkboard and have the pupils assist in writing the sum and a multiplication sentence for each:

(a)	(b)	(c)	(d)
10	12		
10	12	98	109
10	12	98	109
10	12	98	109
+ 10	+ 12	+ 98	+ 109
50	60	392	436
5 x 10 = 50	5 x 12 = 60	4 x 98 = 392	4 x 109 = 436

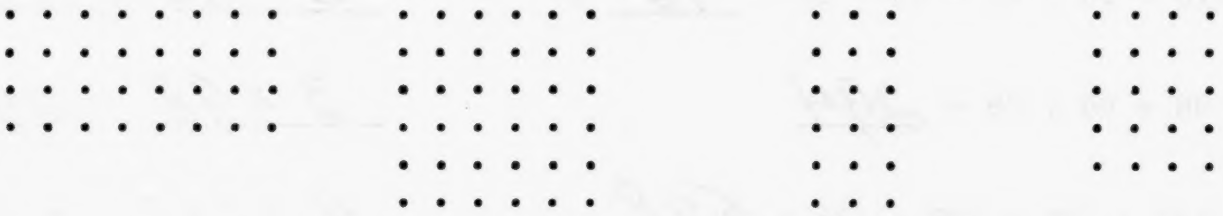
Next, have pupils illustrate arrays for each of the following and find the missing numeral for each. Write these on the chalkboard:*

(a) $4 \times 8 = \boxed{32}$

(b) $6 \times 6 = \boxed{36}$

(c) $\boxed{6} \times 3 = 18$

(d) $5 \times \boxed{4} = 20$



On the chalkboard, write the following number sentences and have the pupils recite a possible number story. For example:

	Number Sentence	Story
(a)	3 x 4 = 12	Bill and Bob found 3 birds' nests in the canyon near Linda Vista. In each nest there were 4 eggs.
(b)	5 x 3 = 15	Alice gathered 3 maple leaves, 3 apple leaves, 3 willow leaves, 3 oak leaves, and 3 cottonwood leaves.

Pupil Exercises. Provide each pupil with worksheet No. 16.

Discuss the worksheet before independent work is assigned. With the less able group, offer as much help as you feel is needed. Problems 9 and 10 may be done cooperatively with the group.

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* On the chalkboard x's or small o's should be used since they are easily distinguished from a dot.

FIND THE SUM, THEN WRITE A MULTIPLICATION SENTENCE FOR EACH EXAMPLE.

1. 10 + 10 + 10 + 10 = 40 4 x 10 = 40
2. 15 + 15 + 15 + 15 + 15 = 75 5 x 15 = 75
3. 98 + 98 + 98 = 294 3 x 98 = 294
4. 132 + 132 + 132 + 132 = 528 4 x 132 = 528
5. 250 + 250 + 250 + 250 = 1000 4 x 250 = 1000

DRAW AN ARRAY FOR EACH MULTIPLICATION SENTENCE, THEN FIND THE PRODUCT.

6. (a) 3 × 5 = 15

X X X X X
X X X X X
X X X X X

(b) 4 × 6 = 24

X X X X X X
X X X X X X
X X X X X X
X X X X X X

(c) 5 × 7 = 35

X X X X X X X
X X X X X X X
X X X X X X X
X X X X X X X
X X X X X X X

FIND THE PRODUCT OF EACH.

7\.

(a)	(b)	(c)
96 × 5 = 480	28 × 4 = 112	215 × 3 = 645
83 × 5 = 415	98 × 4 = 392	54 × 5 = 270
79 × 4 = 316	4 × 98 = 392	3 × 215 = 645

8. Mark and Gregory kept a record of robins, sparrows, and bluejays they saw on the way to school. They saw 9 of each kind of bird. How many birds did their record show?

3x9 = □; 3x9 = 27

9. In Room 21 there are 6 rows of desks with 7 desks in each row. How many desks are in Room 21?

6x7 = □; 6x7 = 42

2. Commutative Principle of Multiplication

Objective. To help pupils understand and use the commutative principle of multiplication which is exemplified by examples such as: 5 × 7 = 7 × 5, 6 × 4 = 4 × 6, etc. These examples illustrate that the order of the factors does not affect the product.

Background Information for the Teacher. Multiplication and addition have certain properties in common: (a) Both are binary operations, that is, only two numbers can be operated upon at one time. (b) Both are commutative. An example of the commutative principle of addition is 3 + 8 = 8 + 3, and an example of the commutative principle of multiplication is 5 × 4 = 4 × 5. In the addition example the sum did not change; in the multiplication example, the product did not change.

Multiplication of two numbers may be thought of in various ways. We may think of them as the number to be multiplied (or repeatedly added) and the number of times it is to be multiplied (or added).

TEACHING PROCEDURE

Exploration. Review with the pupils the commutative principle of addition. Next, write the following examples on the chalkboard and ask the pupils to analyze them in terms of their effect on the product.

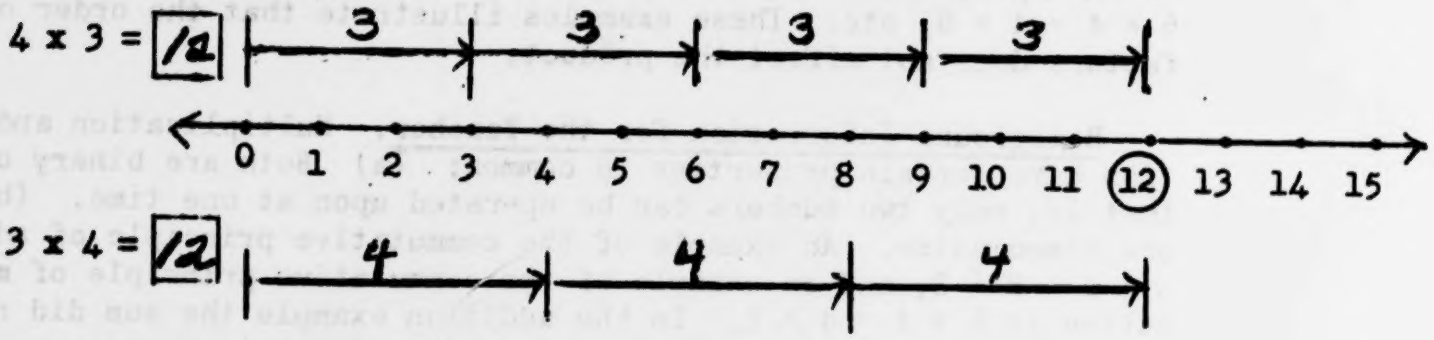
(a)	(b)
2 × 5 = 10	5 × 2 = 10
5 × 6 = 30	6 × 5 = 30
4 × 6 = 24	6 × 4 = 24
7 × 5 = 35	5 × 7 = 35

Ask the pupils, "In each corresponding example, what can we say about the product? The product in each example is the same.

How does changing the order of the factors affect the product? The product remains the same. What statement can we make regarding the change in the order of factors which will always hold true?"

In multiplication changing the order of the factors does not affect the product.

Using the same number line ask someone to come to the chalkboard to show that 4 x 3 = 3 x 4. The illustration might look like this:



Ask the following questions:

1. What happens when we change 4 × 3 to 3 × 4 on the number line?

   The product is the same but the steps are different.

2. What does 3 × 4 (product expression) tell us? That there are 3 sets, each having 4 members.

3. Does the product expression 4 × 3 differ from the product expression 3 × 4? Yes. 4X3 shows that there are 4 sets with 3 members in each set; 3X4 shows that there are 3 sets, each having 4 members.

4. Does the order of the factors affect the product? No.

Use other examples such as 5 × 4, 9 × 3, 10 × 2, etc. and ask questions similar to those listed above.

While some of the pupils work at the chalkboard, have others work at their desks. Pupils should be encouraged to construct their own number lines.

3. Associative Principle of Multiplication

Objectives. (1) To help pupils understand that in multiplication various groupings are permissible. (2) To help pupils understand that multiplication is a binary operation; namely, just two numbers can be multiplied at one time.

Background Information for the Teacher. In multiplying three or more numbers, the factors may be grouped or paired together in more than one way without affecting the product. For example, in multiplying 3 x 4 x 5, we may group the factors as (3 x 4) x 5 or as 3 x (4 x 5). In either case, the product is 60. This property is named the associative principle of multiplication.

In multiplication it is sometimes necessary to find products involving more than two factors, for example, $4 \times 2 \times 3 = \square$. Since the operation of multiplication is associative, the number sentence can be solved.

As in addition, parentheses are used to state which operation is to be performed first. We shall also use it for indicating what numbers belong together.

Because of the associative principle of multiplication and the commutative principle of multiplication, we can group the factors and multiply them in any order. In the number sentence $9 \times 2 \times 5 = \square$, we can multiply from left to right or from right to left. For example:

$$(9 \times 2) \times 5 = 18 \times 5 = 90$$

or

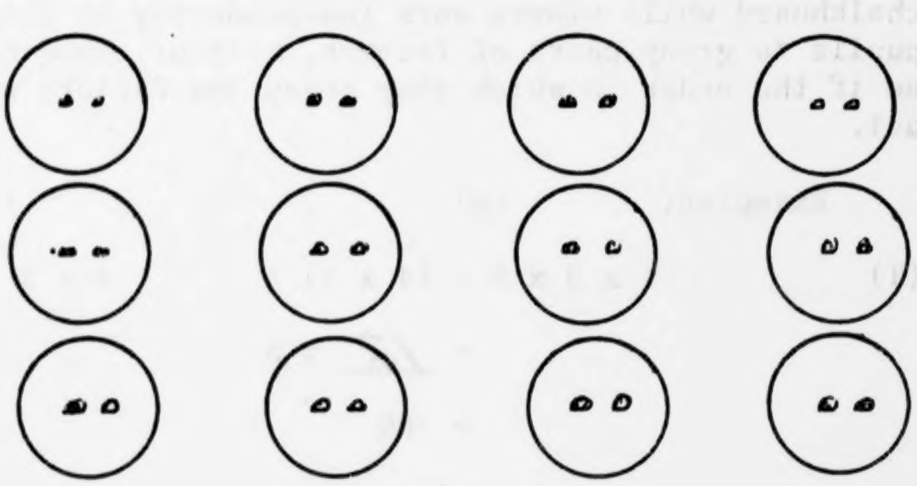
$$9 \times (2 \times 5) = 9 \times 10 = 90$$

or

$$(9 \times 5) \times 2 = 45 \times 2 = 90$$

TEACHING PROCEDURE

Exploration. To provide pupils with a graphic illustration of the associative principle of multiplication, construct the following example on the chalkboard. This illustration represents 3 sets of buttons with 4 buttons in each set and each button has 2 holes.



Ask the following questions:

(1) "How many sets of buttons do we have? 3. How many are in each set?" 4

(2) "How many holes are in each button?" 2

(3) "Is it possible to illustrate this chart in multiplication sentence form?" Yes

(4) "Who would come to the chalkboard and write the sentence and name each numeral?"

3	x	4	x	2	=	24
sets		members		holes		product

(5) "Show with parentheses which two numbers we are going to multiply first?" (3 x 4) x 2

(6) "What does each numeral in the sentence (3 x 4) x 2 tell us?"
The (3 x 4) tells us the number of members or buttons in the array; the 2 tells us the number of holes in each member.

(7) "What is the product for the multiplication sentence (3 x 4) x 2 = $\boxed{24}$?"

(8) "If we grouped (4 x 2) so that our sentence would look like this: 3 x (4 x 2) = $\boxed{24}$, would the product change?" No.

(9) "What does each numeral in the equation 3 x (4 x 2) = $\boxed{24}$ tell us?

3	x	(4 x 2)	=	$\boxed{24}$
sets		holes in each set		product

(10) "What can we say about the order of factors when we multiply?"
The order does not affect the product.

To reinforce the associative principle of multiplication, write several practice examples on the chalkboard. Have some pupils come to the chalkboard while others work independently at their desks. Direct the pupils to group pairs of factors, without changing the arrangement, to see if the order in which they group the factors will affect the product.

Examples:	(a)	(b)
(1)	4 x 3 x 5 = (4 x 3) x 5	4 x 3 x 5 = 4 x (3 x 5)
	= 12 x 5	= 4 x 15
	= 60	= 60

	(a)	(b)
(2)	5 × 2 × 4 = (5 × 2) × 4	5 × 2 × 4 = 5 × (2 × 4)
	= 10 × 4	= 5 × 8
	= 40	= 40
(3)	6 × 1 × 3 = (6 × 1) × 3	6 × 1 × 3 = 6 × (1 × 3)
	= 6 × 3	= 6 × 3
	= 18	= 18

After sufficient practice with a variety of examples, lead the pupils to generalize that "the order of the factors does not affect the product." Use of the phrase "associative principle of multiplication" is optional with fourth-grade pupils.

Pupil Exercises. Provide each pupil with worksheet No. 17.

Limit the explanation for this worksheet with the more able group. However, with the less able, offer as much help as is needed, especially with the "story problems."

COMPLETE THE FOLLOWING:

1. When adding numbers, the order in which we group the addends does not affect the sum.

2. When multiplying numbers, the order in which we group the factors does not affect the product.

WRITE THE MISSING NUMERAL IN EACH BLANK SPACE.

	(a)	(b)
3.	(3 × 3) × 2 = 9 × 2 = 18	3 × (3 × 2) = 3 × 6 = 18
4.	(4 × 5) × 4 = 20 × 4 = 80	4 × (5 × 4) = 4 × 20 = 80
5.	(5 × 6) × 6 = 30 × 6 = 180	5 × (6 × 6) = 5 × 36 = 180
6.	(23 + 17) + 30 = 40 + 30 = 70	23 + (17 + 30) = 23 + 47 = 70
7.	(18 + 12) + 43 = 30 + 43 = 73	18 + (12 + 43) = 18 + 55 = 73
8.	(9 × 1) × 2 = 9 × 2 = 18	9 × (1 × 2) = 9 × 2 = 18
9.	6 × (2 × 0) = 6 × 0 = 0	(6 × 2) × 0 = 12 × 0 = 0
10.	(9 × 6) × 5 = 54 × 5 = 270	9 × (6 × 5) = 9 × 30 = 270

SOLVE EACH PROBLEM. SHOW THAT GROUPING THE ADDENDS IN ADDITION DOES NOT AFFECT THE SUM; THAT GROUPING THE FACTORS DOES NOT AFFECT THE PRODUCT.

11. Several children had a ride on the Ferris wheel at the school carnival. There were 2 children in each of 16 seats. How many children were riding on the Ferris wheel?

2 × 16 = 32; 16 × 2 = 32

12. The ticket salesman sold 115 tickets for the merry-go-round ride in the morning. In the afternoon he sold 138 tickets. That evening he sold 262 tickets. How many tickets did he sell that day?

(115 + 138) + 262 = 515; 115 + (138 + 262) = 515

4. Distributive Principle of Multiplication OVER addition

Objective. To assist pupils in understanding the algorithm of multiplication.

Background Information for the Teacher. The distributive principle* is a fundamental property in the multiplication algorithm. An example of this property is: $3 \times (10 + 7) = (3 \times 10) + (3 \times 7)$. This principle differs from the commutative and associative principles in that it always involves two of the four basic operations. The distributive principle shows the relationship between the operations of multiplication and addition. According to this principle, to find the value of $3 \times (4 + 5)$, we can either (a) add the expression $(4 + 5)$ and then multiply the sum by 3 or (b) multiply both 4 and 5 by 3 and then add the products.

Computation in multiplication can be simplified and the multiplication facts, which have been temporarily forgotten, can be reconstructed if the pupil understands the distributive principle of multiplication. This principle can be illustrated by way of the following:

$$\begin{array}{rcr} \text{(a)} & & \text{(b)} \\ 7 & = & 4 + 3 \\ \times\ 5 & = & \times \qquad 5 \\ \hline 35 & = & 20 + 15 \ = \ 35 \end{array}$$

In illustration (a) we have the usual algorithm or conventional model; the product is 35. In illustration (b) we renamed 7 as $(4 + 3)$ and multiplied; the product is the same.

Often in multiplication the work is simplified if a two-digit number is expressed as a sum of tens and ones, such as $5 \times 35 = 5 \times (30 + 5)$.

Since addition is commutative, we can also express $5 \times (30 + 5)$ as $5 \times (5 + 30)$. Observe how the commutative principle of multiplication and the distributive principle are used in the solution for illustration (e):

(c)

$$\begin{array}{r} 35 \\ \times\ \ 5 \\ \hline 25 \\ 150 \\ \hline 175 \end{array}$$

(d)

$$\begin{array}{r} 30 + 5 \\ \times \qquad 5 \\ \hline 150 + 25 = 175 \end{array}$$

(e)

$$\begin{aligned} 5 \times 35 &= 5 \times (30 + 5) \\ &= (5 \times 30) + (5 \times 5) \end{aligned} \Bigg] \text{ distributive principle}$$

$$\begin{aligned} &= 150 + 25 \\ &= 175 \end{aligned}$$

The only difference between the vertical and the horizontal form is that in the vertical form, illustration (c), 5×5 is done before 5×30, illustration (d).

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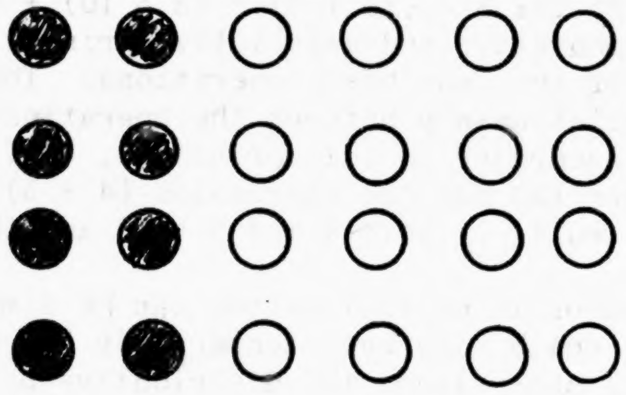
* See Glossary, page 134.

TEACHING PROCEDURE

Exploration. On the chalkboard represent a 4 × 6 array such as the illustration below:



Ask and state the following which may help to make a discovery:

1. How many objects are represented in this illustration? 24
2. Write a multiplication sentence for this entire illustration. 4X6=24
3. Write a multiplication sentence for those objects which are shaded. 4X2=8
4. Write a number sentence for those objects which are not shaded. 4X4=16
5. If we add the product of 4 × 2 which is 8 and the product of 4 × 4 which is 16, what will be the sum? 8+16=24 How does the product of 4 × 6 = ☐ compare to the product of 4 × 2 = ☐ plus 4 × 4 = ☐ ?

The products are the same.

Now, let's examine the following number sentence examples:

(a) 4 × 6 = 24

(b) 4 × 6 = 4 × (2 + 4) = 24

(c) 4 × 6 = (4 x 2) + (4 × 4) = 24

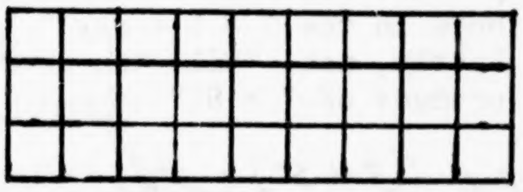
Ask the pupils to make some observations, guiding them to state, for example:

(1) Example (a) has 4 sets and 6 members in each set.

(2) Example (b) renames 6 as (2+4).

(3) Example (c) represents 4 sets of 2 plus 4 sets of 4.

To provide additional understanding of the distributive principle of multiplication over addition, illustrate the following:



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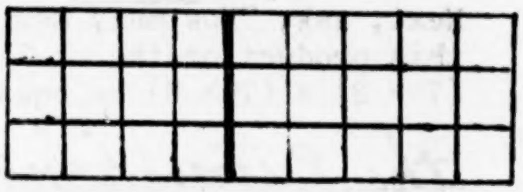
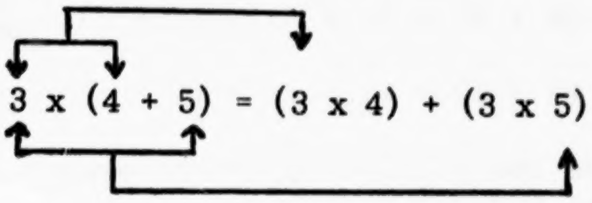


3 x 9 = (3 x 4) + (3 x 5)

= 12 + 15

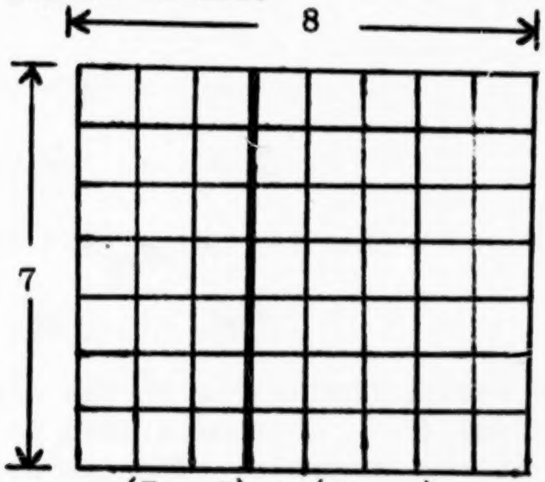
27 = 27

Help the pupil discover that the distributive principle of multiplication over addition allows us to use the operations of multiplication and addition in the manner shown in the above example. The 3 x 9 array is renamed as a 3 x 4 array plus a 3 x 5 array. The 3 distributes over 4 and over 5, for example:



3 x (4 + 5) = (3 x 4) + (3 x 5)

Continue by having pupils interpret several more illustrations in the same way. Next, make an array, as illustrated below, from a piece of tagboard that can be folded.



(7 x 3) + (7 x 5)

Show the array to the class and ask, "How many rows and how many columns are in this array? 7 rows and 8 columns What is the product of this 7 x 8 array?" 56

Now fold the card to show that the 7 × 8 array can be thought of as a 7 × 3 array plus a 7 × 5 array. Ask, "How many members are there in the 7 × 3 array?" 21 (Have this product recorded on the chalkboard.) Next, ask, "How many members are there in the 7 × 5 array?" 35 (Record this product on the chalkboard.) Lastly, ask, "Will the product of (7 × 3) + (7 × 5) be equal to the product of 7 × 8?" yes, because the 8 was renamed as (3+5); the rows and columns did not change.

Continue to illustrate the distributive property of multiplication over addition by folding the same card, 7 × 8 array, in other ways. For example: 7 × (6 + 2); 7 × (7 + 1); and 7 × (4 + 4). Have the related number sentences recorded each time.

For individual practice at their desks and at the chalkboard, write these and similar examples on the chalkboard to clarify the idea that we get the same result whether we add first and then multiply, or whether we multiply first and then add.

(a) $6 \times (4 + 2) = (6 \times 4) + (6 \times 2) = \underline{24} + \underline{12} = \underline{36}$

(b) $5 \times (3 + 6) = (5 \times 3) + (5 \times 6) = \underline{15} + \underline{30} = \underline{45}$

Next, have the pupils copy these examples from the chalkboard and put in the missing numerals.

(a) $3 \times 8 = (3 \times 3) + (3 \times 5)$

$= \underline{9} + \underline{15} = \boxed{24}$

(b) $4 \times 7 = (4 \times 6) + (4 \times 1)$

$= \underline{24} + \underline{4} = \boxed{28}$

(c) $4 \times 6 = (4 \times 3) + (4 \times 3)$

$= \underline{12} + \underline{12} = \boxed{24}$

When the pupils become familiar with the distributive principle of multiplication over addition, introduce them to its use when multiplying numbers named by two-place numerals. The emphasis should be on understanding what really happens when we multiply.

For example, illustrate on the chalkboard three ways in which the following example may be performed:

(a)

$$\begin{array}{r} 76 \\ \times\ \ 4 \\ \hline 304 \end{array}$$

(b)

$$\begin{array}{r} 76 \\ \times\ \ 4 \\ \hline 24 \\ 280 \\ \hline 304 \end{array}$$

(c)

$$\begin{aligned} 4 \times 76 &= 4 \times (70 + 6) \\ &= (4 \times 70) + (4 \times 6) \\ &= 280 + 24 \\ &= 304 \end{aligned}$$

Ask, "Which example shows what is really happening when we multiply a two-place numeral by a one-place numeral?" (c), (b)

"Which example appears to conceal the meaning by making it a mechanical operation?" (a)

Provide the pupils with practice work by writing these examples on the chalkboard. Have them copy each example and put in the missing numerals.*

(a) $9 \times 21 = (9 \times 20) + (9 \times \underline{1})$

$= \underline{180} + \underline{9}$ or $\boxed{189}$

(b) $5 \times 54 = (5 \times \underline{50}) + (5 \times 4)$

$= \underline{250} + \underline{20}$ or $\boxed{270}$

(c) $8 \times 23 = (8 \times 20) + (8 \times 3)$

$= \underline{160} + \underline{24}$ or $\boxed{184}$

(d) $6 \times 45 = (6 \times 40) + (6 \times 5)$

$= \underline{240} + \underline{30}$ or $\boxed{270}$

Multiplying hundreds, tens, and ones (three-place numeral) using the distributive principle of multiplication over addition operates the same way as it does for multiplying a two-place numeral. Point out to the pupils that the result of 3 × 623 may be achieved in several ways. There is the "short-cut" method (a), the "repeated addition" method (b), and "structural methods" (c), (d), and (e). Direct the pupils to examine each of the following: **

(a)

$$\begin{array}{r} 623 \\ \times\ \ 3 \\ \hline 1869 \end{array}$$

(b)

$$\begin{array}{r} 623 \\ 623 \\ +\ 623 \\ \hline 1869 \end{array}$$

(c)

$$\begin{array}{rl} 623 = 600 + 20 + 3 & \\ \times \qquad\qquad\quad 3 & \\ \hline 1800 + 60 + 9 & = \\ (1800 + 60) + 9 & = \\ 1860 + 9 & = 1,869 \end{array}$$

---

* All of these examples need not be used; use only those which you feel are necessary to provide the needed practice.

** All of these examples need not be used; use only those which you feel are appropriate for your pupils.

(d)

$$\begin{aligned} 3 \times 623 &= 3 \times (600 + 20 + 3) \\ &= (3 \times 600) + (3 \times 20) + (3 \times 9) \\ &= 1800 + 60 + 9 \\ &= (1800 + 60) + 9 \\ &= 1860 + 9 \\ &= 1869 \end{aligned}$$

(e)

$$\begin{array}{rl} 623 & \\ \times\ \ \ 3 & \\ \hline 9 & = (9 \times 1) \\ 60 & = (3 \times 20) \\ +\ 1800 & = (3 \times 600) \\ \hline 1869 & \end{array}$$

Now ask, "Which example conceals the meaning of what takes place when we multiply? Example (a) Which example would be 'too long' if we were to show the product of 38 × 432 by repeated addition? Example (b) Which examples have more 'meaning' and actually show what is happening when we multiply?" Examples (c), (d), and (e).

For individual practice, have the pupils copy from the chalkboard each of the following examples and put in the missing numerals:

(a)

$$\begin{aligned} 3 \times 231 &= 3 \times (200 + 30 + 1) \\ &= (3 \times 200) + (3 \times \underline{30}) + (3 \times 1) \\ &= \underline{600} + \underline{90} + \underline{3} \text{ or } \boxed{693} \end{aligned}$$

(b)

$$\begin{aligned} 4 \times 122 &= 4 \times (100 + 20 + \underline{2}) \\ &= (4 \times 100) + (4 \times \underline{20}) + (4 \times 2) \\ &= 400 + \underline{80} + \underline{8} \text{ or } \boxed{488} \end{aligned}$$

(c)

$$\begin{aligned} 8 \times 322 &= 8 \times (\underline{300} + \underline{20} + \underline{2}) \\ &= (8 \times 300) + (8 \times 20) + (8 \times 2) \\ &= \underline{2400} + \underline{160} + \underline{16} \text{ or } \boxed{2576} \end{aligned}$$

Pupil Exercises. Provide each pupil with worksheet No. 18.

Limit the explanation of this worksheet with the more able pupils. With the less able group, provide sufficient practice and assistance before independent work is assigned.

Interim Mathematics Unit - Grade 4
San Diego City Schools (1-65)

Worksheet No. 18

FOR EACH OF THE FOLLOWING EXAMPLES, FILL IN THE MISSING NUMERALS:

(a)

1. 8 × 10 = 8 × (6 + 4)
= (8 × 6) + (8 × 4)
= 48 + 32
= 80

(b)

6 × 10 = 6 × (3 + 7)
= (6 × 3) + (6 × 7)
= 18 + 42
= 60

2. 10 × 9 = 10 × (5 + 4)
= (10 × 5) + (10 × 4)
= 50 + 40
= 90

12 × 6 = 12 × (3 + 3)
= (12 × 3) + (12 × 3)
= 36 36
= 72

3. 8 × 23 = (8 × 20) + (8 × 3)
= 160 + 24
= 184

5 × 24 = (5 × 20) + (5 × 4)
= 100 + 20
= 120

4. 6 × 314 = (6 × 300) + (6 × 10) + (6 × 4)
= 1800 + 60 + 24 = 1,884

5. 3 × 633 = (3 × 600) + (3 × 30) + (3 × 3)
= 1800 + 90 + 9 = 1,899

6. 6 × 311 = (6 × 300) + (6 × 10) + (6 × 1)
= 1800 + 60 + 6 = 1,866

(a) (b) (c)

7. 7 × 80 = △ 560 9 × 50 = □ 450 6 × 70 = □ 420

8. 6 × 110 = □ 660 9 × 35 = △ 315 8 × 70 = □ 560

DRAW A CIRCLE AROUND THE LETTER FOR EACH TRUE SENTENCE.

9. [✓ a.] 5 × 9 = (5 × 7) + (5 × 2)
[✓ b.] 6 × 17 = (6 × 10) + (7 × 6)
[✓ c.] 9 × 12 = (9 × 6) + (9 × 6)
[✓ d.] 15 < (8 × 2)

5. Division and Multiplication Relationship

Objective. To review and extend the meaning of division using sets, arrays, and number sentences.

Background Information for the Teacher. Division of whole numbers is defined as a binary operation in which two given whole numbers are combined to produce a third whole number called the quotient. There are examples, however, when it is not possible to produce a third whole number.

Just as multiplication is thought of as repeated addition, division is sometimes thought of as repeated subtraction. Consider for a moment the example, $12 \div 3 = \square$. The result of dividing the pair of numbers 12 and 3 begins with 12. Subtracting 3 from 12 leaves 9; subtracting 3 from 9 leaves 6; subtracting 3 from 6 leaves 3; etc. We continue to subtract 3 until we get a difference of 0. The number of times that 3 has been subtracted from 12 is the result of $12 \div 3$, in this case, 4 (see illustration).

$$\begin{array}{r l} 12 & \\ \underline{-\ 3} & 1 \\ 9 & \\ \underline{-\ 3} & 1 \\ 6 & \\ \underline{-\ 3} & 1 \\ 3 & \\ \underline{-\ 3} & \underline{1} \\ 0 & 4 \end{array}$$

This discovery permits us to design an algorithm to speed up the search for the quotient. Thus, we may arrange our numerals as follows:

$$\begin{array}{r r|l} 3) & 12 & \\ & \underline{-\ 3} & 1 \\ & 9 & \\ & \underline{-\ 3} & 1 \\ & 6 & \\ & \underline{-\ 3} & 1 \\ & 3 & \\ & -\ 3 & 1 \\ \hline & 0 & 4 \end{array}$$

When the pupil knows his multiplication combinations well and has had an opportunity to analyze many examples, he will recognize at once that 3 will be subtracted repeatedly, 4 times. Thus, he may use this algorithm:

$$\begin{array}{r} 4 \\ 3\overline{)\,12} \\ \underline{-12} \\ 0 \end{array}$$

If we consider the pair of numbers 36 and 5, we will recognize that the quotient is not a whole number but a fractional number, $7\frac{1}{5}$, as shown in the illustrations (a) and (b) below:

(a)

$$36 \div 5 = 7\frac{1}{5}$$

(b)

$$\begin{array}{r} 7 \\ 5\overline{)\,36} \\ -\,35 \\ \hline 1 \end{array}$$

Interpreting division as repeated subtraction does not always give us a quotient which is in the set of whole numbers. However, the repeated subtraction concept does promote the understanding of the division algorithm.

To study the operation of division, let's look to multiplication to see if a relationship exists. For example, by examining illustrations (a) and (b) below:

(a)

Factor		Factor		Product
3	×	4	=	12
4	×	3	=	12

(b)

Product		Factor		Factor
12	÷	3	=	4
12	÷	4	=	3

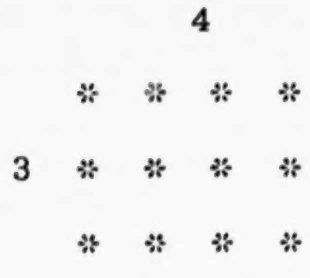
We can see that the "known" factors in multiplication can satisfy the "missing factors" in division. For example:

If $3 \times 4 = 12$, then $12 \div 3 = 4$ because $3 \times 4 = 12$;

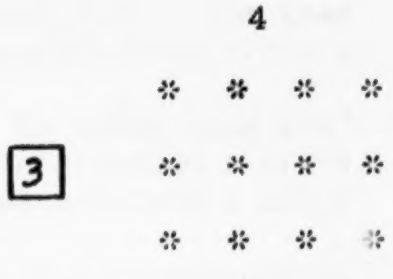
If $3 \times 4 = 12$, then $12 \div 4 = 3$ because $4 \times 3 = 12$.

Using an array, we might represent 12 objects as 3 sets with 4 members in each set, as in illustration (a):

(a)



(b)

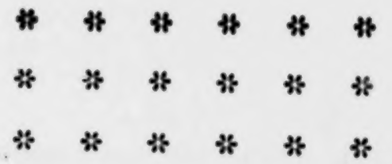


In illustration (b) we know that there are 12 objects, one factor is "known," that is, 4; we are looking for the "missing factor" which is 3.

TEACHING PROCEDURE

Exploration. On the chalkboard write two related multiplication sentences, such as 5 × 6 = 30 and 6 × 5 = 30. Now ask, "If five 6's are equal to 30, how many 6's are in 30? 5. If six 5's are equal to 30, how many 5's are in 30?" 6.

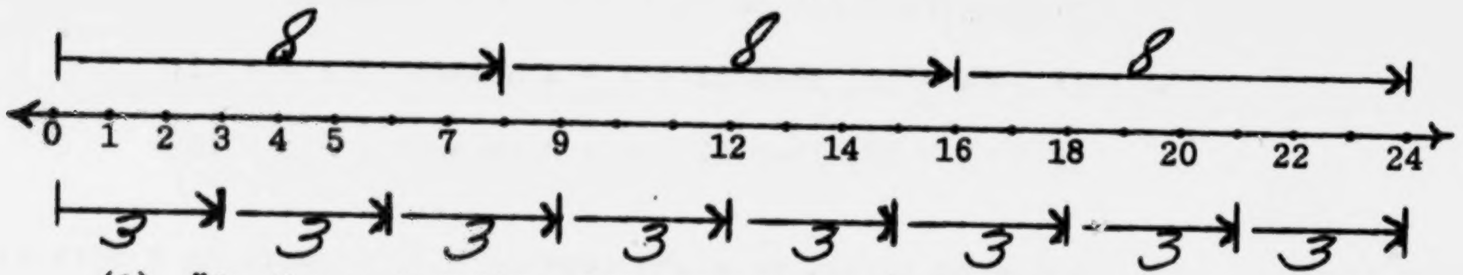
Next, illustrate a 3 × 6 array and ask, "How many sets of 3 are



there in 18? 6. How many sets of 6 are there in 18? 3. What is the product of 3 × 6 = $\boxed{18}$? Name the missing factor in 18 ÷ 3 = $\boxed{6}$." Next, write the two multiplication sentences and the two division sentences for the 3 × 6 array shown above.

Multiplication	Division
3 × 6 = 18	18 ÷ 3 = 6
6 × 3 = 18	18 ÷ 6 = 3

Use the number line to increase understanding between multiplication and division. Illustrate the following on the chalkboard and have pupils identify the measure of each segment, then ask the following questions:



(1) "On the number line, how many moves of 8 lengths are shown? 3
Write a multiplication sentence to show three 8's." 3 X 8 = 24

(2) "If we begin at 24 on the number line and move to the left, how many moves of 8 lengths can we make to get back to 0 ? 3 Write a division sentence to show this." 24 ÷ 8 = 3

(3) "How many moves of 3 lengths are shown on the number line?" 8
Write a multiplication sentence to show eight 3's." 8 X 3 = 24
Write a division sentence to show this." 24 ÷ 3 = 8

On the chalkboard write the following examples for individual practice. Inform the pupils that the four sentences in each row are related; knowing the solution can be helpful in solving the others.

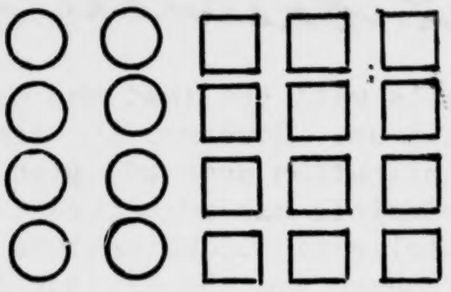
1. 6 × 3 = 18 18 ÷ 3 = 6 3 × 6 = 18 18 ÷ 6 = 3

2. 2 × 8 = 16 16 ÷ 8 = 2 8 × 2 = 16 16 ÷ 2 = 8

3. 14 ÷ 7 = 2 2 × 7 = 14 14 ÷ 2 = 7 7 × 2 = 14

Before introducing division and addition to discover some new facts, review the distributive principle of multiplication over addition. Since multiplication and division are related, separating a factor into two addends will be employed in division.

On the chalkboard, illustrate a 4 × 5 array such as the one below.*



Now ask the pupils, "How many objects are represented in each row? 5 How many objects are represented in each column? 4 Write a multiplication sentence to show how many objects are represented in this complete array. 4 × 5 = 20 Is it possible to show a division sentence using the numbers 4, 5, and 20? Yes. 20 ÷ 5 = 4; 20 ÷ 4 = 5. Does the sentence 20 ÷ 4 = ☐ tell us something about the illustration? Yes, we are looking for the missing factor which is 5.

Next, guide the pupils to find the missing factor in the following by asking:

1. How many circles are represented in our illustration? 8 How many rows of circles are there? 4 To find the number of members in each row of circles (the missing factor), what must we do? Divide; 8 ÷ 4 = 2; there are 2 members in each set of circles.

2. How many squares are represented in our illustration? 12 How many rows of squares are there? 4 To find the number of members in each row of squares (the missing factor), what must we do? Divide; 12 ÷ 4 = 3; there are three members in each set of squares.

---

* The use of the flannel board and cutouts may be helpful for this concept. For example, the teacher may wish to use 4 rows of geometric shapes: 2 columns of circles and 2 columns of squares.

3. "Does our illustration show that we have 8 circles and 12 squares? yes Is (8 + 12) another name for 20? yes Then, what can we say about this sentence?"

$$(8 + 12) \div 4 = (8 \div 4) + (12 \div 4)$$

This sentence is true; 20 is renamed so that we can divide twice which is easier than dividing once, especially if you do not know the combinations.

It is unlikely that the pupils will see that the divisor is distributed to 8 and 12 with respect to addition. However, the similarity between the distributive principle of multiplication over addition and that of division should be pointed out. This principle has advantages; if the dividend is too large and the division-multiplication combinations are not completely mastered, the dividend may be renamed and the divisor distributed.

Since this is an advanced concept, many illustrations and examples will have to be presented, proceeding from the concrete (objects) to the semi-abstract (symbols).

Next, point out to the pupils that in order to find the missing factor in this example, $25 \div 5 = \square$, it is possible to choose another name for 25 which is convenient for simplifying division. For example, 25 may be renamed as: (15 + 10), (20 + 5), (10 + 15), etc. On the chalkboard, illustrate the following:

$$25 \div 5 = \square$$

$$(15 + 10) \div 5 = (15 \div 5) + (10 \div 5)$$

$$= \underline{\ 3\ } + \underline{\ 2\ }, \text{ or } \boxed{5}$$

To analyze what has actually taken place, ask these and similar questions: *

1. "How have we renamed 25? (15+10) How many times have we divided? Twice: (15÷5) and (10÷5) What is the missing factor for $15 \div 5 = \square$? 3 What is the missing factor for $10 \div 5 = \square$? 2 What is the missing factor for $25 \div 5 = \square$?" 5
2. "How does the missing factor for $25 \div 5 = \square$ compare with the sum of the missing factors for $(15 \div 5) + (10 \div 5)$? They are equal. Does $25 \div 5 = (15 \div 5) + (10 \div 5)$?" yes.

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* Record the pupils' responses in each blank space in the illustration.

"Would renaming 25 as (13 + 12) be convenient to simplify division?"

No, because (13 ÷ 5) + (12 ÷ 5) would not name whole numbers.

For pupil practice, write the following examples on the chalkboard. Have some pupils work at the chalkboard while others work at their desks. Lead them to increase their understanding of the distributive idea by having them fill in the missing numerals.

(a) $30 \div 5 = \boxed{6}$; $30 \div 6 = \boxed{5}$; $5 \times 6 = \boxed{30}$;
$6 \times 5 = \boxed{30}$

(b) $\boxed{6} \times 5 = 30$; $6 \times \boxed{5} = 30$; $30 \div \boxed{6} = 5$; $30 \div \boxed{5} = 6$

(c) $28 \div 4 = (20 \div 4) + (8 \div 4)$
$= \underline{5} + \underline{2}$, or $\boxed{7}$

(d) $24 \div 3 = (18 \div 3) + (6 \div 3)$
$= \underline{6} + \underline{2}$, or $\boxed{8}$

(e) $30 \div 5 = (15 \div 5) + (15 \div 5)$
$= \underline{3} + \underline{3}$, or $\boxed{6}$

Pupil Exercises. Provide each pupil with worksheet No. 19.

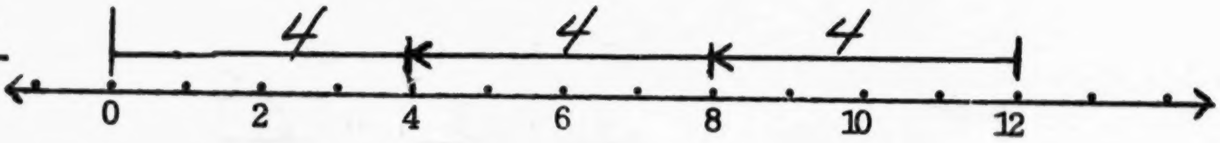
The pupils will need a ruler or straightedge to do examples 5 through 10. Encourage neat and accurate work. With less able pupils, some of these examples may be worked cooperatively at the chalkboard.

WRITE THE MISSING NUMERALS IN THE BLANK SPACES FOR EACH OF THE FOLLOWING:

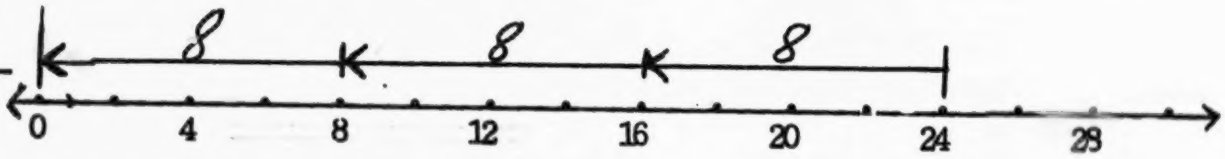
	(a)	(b)
1.	6 × 7 = (6 × 4) + (6 × 3) = 24 + 18 or 42	5 × 8 = (5 × 5) + (5 × 3) = 25 + 15 or 40
2.	36 ÷ 4 = (20 ÷ 4) + (16 ÷ 4) = 5 + 4 or 9	28 ÷ 7 = (21 ÷ 7) + (7 ÷ 7) = 3 + 1 or 4
3.	5 × 64 = (5 × 60) + (5 × 4) = 300 + 20 or 320	4 × 69 = (4 × 60) + (4 × 9) = 240 + 36 or 276
4.	85 ÷ 5 = (40 ÷ 5) + (45 ÷ 5) = 8 + 9 or 17	46 7 = (40 × 7) + (6 × 7) = 280 + 42 or 322

ILLUSTRATE THE FOLLOWING NUMBER SENTENCES ON THE NUMBER LINE: *

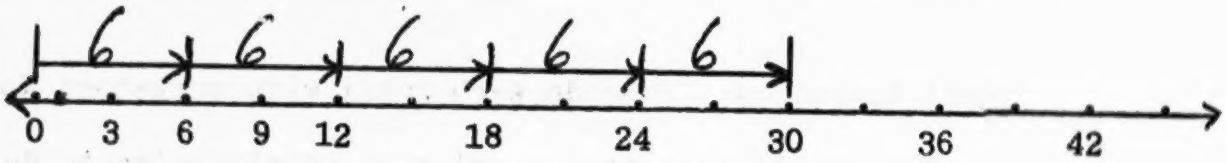
5. 12 ÷ 4 = 3


4
4
4
0
2
4
6
8
10
12

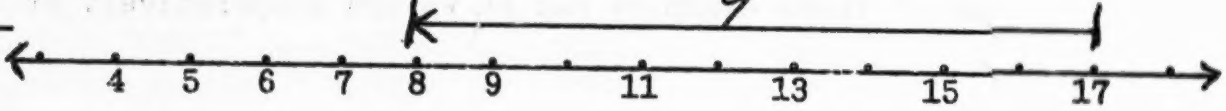
6. 24 ÷ 8 = 3


8
8
8
0
4
8
12
16
20
24
28

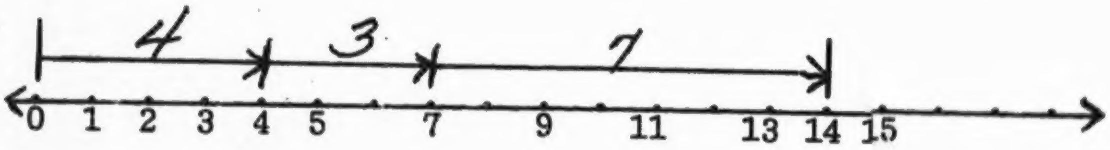
7. 5 × 6 = 30


6
6
6
6
6
0
3
6
9
12
18
24
30
36
42

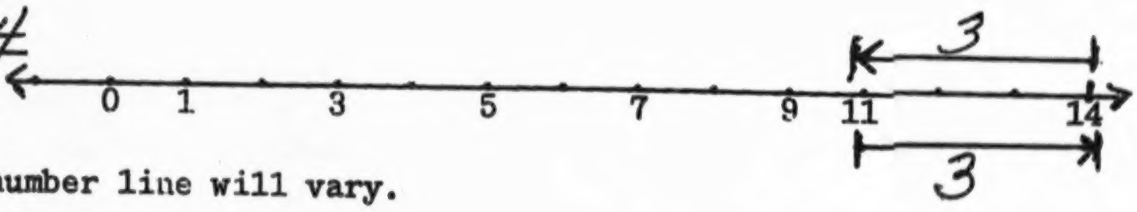
8. 17 - 9 = 8


9
4
5
6
7
8
9
11
13
15
17

9. (4 + 3) + 7 = 14


4
3
7
0
1
2
3
4
5
7
9
11
13
14
15

10. (14 - 3) + 3 = 14


3
0
1
3
5
7
9
11
14
3

*Illustrations on the number line will vary.

6. Dividing Tens and Ones Separately

Objective. To develop skill in renaming a dividend into two parts so that it is convenient to divide the parts separately, using the distributive idea.

Background Information for the Teacher. It is suggested that the teacher present many examples to the class, making certain that pupils understand they are to add the two partial quotients after dividing the parts separately. For example, the conventional algorithm, illustration (a), can be somewhat mechanical; illustration (b) renames 96 as 9 tens and 6 ones or 90 + 6; illustration (c) uses the horizontal notation, incorporating the distributive principle and adding the partial quotients.

(a)

$$\begin{array}{r} 32 \\ 3\overline{)\,96} \\ \underline{9} \\ 6 \\ \underline{6} \end{array}$$

(b)

$$\begin{array}{l} 30 + 2 = 32 \\ 3\overline{)\,90 + 6} \end{array}$$

(c)

$$\begin{aligned} 96 \div 3 &= \square \\ (90 + 6) \div 3 &= (90 \div 3) + (6 \div 3) \\ &= 30 + 2 \\ &= 32 \end{aligned}$$

It may be necessary, especially with some pupils, to use the place-value chart or bundles of sticks to represent tens and ones to illustrate the renaming of the dividend.

TEACHING PROCEDURE

Exploration. On the chalkboard, provide pupils with examples such as the following which will give practice for renaming the dividend as tens and ones. (Later it will be seen that the specific parts selected will depend on the divisor.)

96 = 90 + 6

86 = 80 + 6

48 = 40 + 8

66 = 60 + 6

Next, write the following problem on the chalkboard:

> "Bob has 96 soldiers in his Battleground game. He wants to put them into 3 boxes, with the same number in each box. How many should he put in each box?"

Ask, "What division sentence should we write for this problem? 96 ÷ 3 = ☐ To find the missing numeral, can we rename 96 as (90 + 6)? Yes Would it be convenient to rename 96 as (92 + 4)?" No, because we are trying to show tens and ones in preparation for division.

Write the following example on the chalkboard and ask the pupils to name the missing numerals:

96 ÷ 3 = ☐

(90 + 6) ÷ 3 = (90 ÷ 3) + (6 ÷ 3)

= 30 + 2, or $\boxed{32}$

As the pupils respond to the following questions, write in the correct responses: Ask, "What numerals do we write in the blank spaces? 30 and 2 What is 96 ÷ 3? 32 How many soldiers should Bob put in each box?" 32

Use another division sentence such as 84 ÷ 2 = ☐. Now ask the pupils to tell a story which fits the example. A possible story might be:

> "Four third-grade classrooms from Hawthorne Elementary School went to Coronado. There were 84 children in all. Two buses were provided for the trip. How many children rode in each bus if each bus had the same number of children?"

Have the pupils respond to questions so that the problem may be stated as:

84 ÷ 2 = ☐

(80 + 4) ÷ 2 = (80 ÷ 2) + (4 ÷ 2)

= 40 + 2, or $\boxed{42}$

Ask, "What numerals do we write in the blank spaces? 40 and 2 What is the sum of (40 + 2) = ☐ ? 42 What is 84 ÷ 2 ? 42 .

Provide the pupils with practice so that they may develop skill in thinking of the dividend as tens and ones, adding the partial quotients, and getting the solution. Write these and similar examples on the board:

(a) 44 ÷ 2 = (40 ÷ 2) + (4 ÷ 2)

= 20 + 2 , or 22

(b) 48 ÷ 4 = (40 ÷ 4) + (8 ÷ 4)

= 10 + 2, or 12

(c) 55 ÷ 5 = (50 ÷ 5) + (5 ÷ 5)

= 10 + 1, or 11

(d) 77 ÷ 7 = (70 ÷ 7) + (7 ÷ 7)

= 10 + 1, or 11

In discussing each example, write in the correct responses as the pupils respond to questions regarding the missing numerals.

Whenever the pupils begin to understand division in terms of thinking about the dividend as tens and ones, extend the concept to include higher decade division. For example, write the following problem on the chalkboard:

> "Sarah has 128 pencils to share with 4 friends. She plans to give each friend the same number of pencils. How many pencils will each friend get?"

Next, ask the pupils if the following sentence tells the story about the problem:

128 ÷ 4 = ☐

(120 + 8) ÷ 4 = (120 ÷ 4) + (8 ÷ 4)

= 30 + 2, or $\boxed{32}$

Ask, "How did we rename 128? (120 + 8) How many tens and ones are represented by (120 + 8)? 12 tens and 8 ones What is (120 ÷ 4)? 30 What is (8 ÷ 4)? 2 What is (128 ÷ 4)? 32 Which of the following are true?"

(a) 128 ÷ 4 is equal to (120 ÷ 4) + (8 ÷ 4) True

(b) 128 = 120 + 8 True

(c) 32 × 4 = (30 × 4) + (2 x 4) True

(d) 32 × 4 = 128 True

(e) 128 ÷ 4 = 32 True

Write the following examples on the chalkboard for independent practice at desks:

(a) $255 \div 5 = (250 \div 5) + (5 \div 5)$

$= \underline{25} + \underline{1}$, or $\underline{26}$

(b) $208 \div 4 = (200 \div 4) + (8 \div 4)$

$= \underline{50} + \underline{2}$, or $\underline{52}$

(c) $128 \div 2 = (120 \div 2) + (8 \div 2)$

$= \underline{60} + \underline{4}$, or $\underline{64}$

Encourage the pupils to look for a pattern in determining tens and ones and to work independently. Do not hesitate to give help. Pupils sometimes seem to lose the idea for a moment, then a word at the right time will put them on the track.

Pupil Exercises. Provide each pupil with worksheet No. 20.

Limit the explanation of this worksheet with the more able group. With the less able group, offer as much explanation as you feel is necessary. Problems 1 through 4 may be worked cooperatively with some pupils.

WRITE A COMPLETE NUMBER SENTENCE FOR EACH PROBLEM. DECIDE WHETHER TO ADD, SUBTRACT, MULTIPLY, OR DIVIDE.

1. Men at a lumber camp used 21 power saws, 14 tractors, and 16 trucks for getting lumber to the sawmill. How many items of equipment were used in their work?

   21 + (14 + 16) = 51

2. One truck hauled 18 loads of logs in one day. There were 6 logs in each load. How many logs did the truck haul that day?

   6 x 18 = (6 x 10) + (6 x 8)
   = 60 + 48 or 108

3. Another truck hauled a total of 72 logs in one day. Six logs were hauled on each load. How many loads did the truck haul?

   72 ÷ 6 = (60 + 12) ÷ 6
   = (60 ÷ 6) + (12 ÷ 6)
   = 10 + 2 or 12

4. One of the trucks was driven 1,342 miles during a one-month period and 1,184 miles the following month. What is the difference in the miles driven during the two months?

   1,342 - 1,184 = 158

WRITE THE MISSING NUMERAL IN THE BLANK SPACES.

5. 255 ÷ 5 = (250 ÷ 5) + (5 ÷ 5)
   = 50 + 1 or 51

6. 459 ÷ 9 = (450 ÷ 9) + (9 ÷ 9)
   = 50 + 1 or 51

7. 168 ÷ 4 = (160 ÷ 4) + (8 ÷ 4)
   = 40 + 2 or 42

8. 497 ÷ 7 = (490 ÷ 7) + (7 ÷ 7)
   = 70 + 1 or 71

9. 856 ÷ 8 = (800 ÷ 8) + (56 ÷ 8)
   = 100 + 7 or 107

10. 648 ÷ 8 = (640 ÷ 8) + (8 ÷ 8)
    = 80 + 1 or 81

7. Introducing the Long-division Form

Objective. To develop a division algorithm which is fundamental to future work and leads to a better understanding of the conventional algorithm.

Background Information for the Teacher. The approach to division in this section will be used consistently throughout this unit until the conventional algorithm is employed. However, as the pupils make further progress in dividing tens and ones separately in this grade and later grades, they will need to learn the familiar algorithm which will eventually be used in higher-decade division.

Interpreting division as repeated subtraction as indicated previously (see page 98), does not always give us a quotient which is in the set of whole numbers. However, the repeated subtraction concept does promote the understanding of the division algorithm as illustrated by the story which follows.

Suppose for a moment that three different pupils worked on the example 136 ÷ 8 = ☐ using the repeated subtraction concept. In analyzing examples (a), (b), and (c) representing the work of each pupil, we would probably state that:

```
      (a)                 (b)                 (c)

8 ) 136             8 ) 136              8 ) 136
   - 40 | 5            - 80 | 10            - 80 | 10
     96 |                56 |                 56 |
   - 40 | 5            - 40 |  5            - 56 |  7
     56 |                16 |                  0 | 17
   - 40 | 5            - 16 |  2
     16 |                 0 | 17
   -  8 | 1
      8 |
   -  8 | 1
      0 | 17
```

Pupil (a) is just beginning his study of the division algorithm; pupil (b) is beginning to develop a greater insight into the division algorithm; and pupil (c) is most familiar with the division algorithm. In all three examples, the pupils understand division. After a good development of the division algorithm, all three pupils should write the familiar algorithm, that is:

```
     17
8 ) 136
   - 80
     56
   - 56
      0
```

TEACHING PROCEDURE

Exploration. On the chalkboard, write the following problem:

"During an art lesson, Sue made 20 paper flowers. She wants to put them in vases so as to decorate the classroom. Each vase should contain 5 flowers. How many vases will she need?"

Next, write the following number sentence on the chalkboard:

20 ÷ 5 = □

Ask, "Does this number sentence tell us about the problem? Yes What does the □ represent?" The missing factor, or the number of vases needed.

Next, ask the pupils if the following illustration represents the problem:

Objects	Columns	Rows
20	5	□

From the illustration it is obvious that there are 4 rows. Ask the pupils, "If the 20 objects represent flowers, the 5 columns represent the number of flowers that will be placed in each vase, What will the rows represent? The number of vases needed. How many vases are needed? Is it true that:

20 ÷ 5 = 4 Yes
20 ÷ 4 = 5 Yes
4 × 5 = 20 Yes
5 × 4 = 20 Yes

Since the pupils may have been introduced to the short form 5) 20 in the basic text, the teacher may proceed from this point by telling the pupils that they can write an example for the problem in question as shown below:

5) 20
= □ × 5

Explain that the example states the 20 is to be divided by 5. In other words, how many 5's are contained in 20. It also states that 20 = □ × 5. Ask the following questions:

1. What numeral is missing in the example 20 = □ × 5? 4

2. Since $20 = \boxed{4} \times 5$, let's write this numeral in our example so that we now have:

$$\begin{array}{r|l} 4 & \\ 5\overline{)20} & \\ 20 & = \boxed{4} \times 5 \end{array}$$

We have used this form to find the number of vases needed for the 20 flowers if 5 flowers were to be contained in each vase.

Proceed with other examples so that pupils will relate their multiplication combinations to that of division.

Use these and similar examples to develop the algorithm:

$$\begin{array}{r|l} 6 & \\ 4\overline{)24} & \\ 24 & = \boxed{6} \times 4 \end{array}$$

Ask, "To find the number of 4's in 24, how can we restate the problem? 24 = ☐ x 4. What is the missing numeral? 6 (Write the missing numeral 6 in the frame.) What is 6 × 4? 24 (Write 24 under the dividend 24 in the example.) What have we learned about the example $4\overline{)24}$?" There are 4 sixes in 24. (Write the numeral 6 above the dividend in the example.)

To show quotients by two-place numerals, use the following example:

"Andy put 50 shells in 5 sets, with the same number of shells in each set. How many shells did he put in each set?"

Ask, "Does this number sentence $50 \div 5 = \boxed{}$ represent the problem? Yes. How do we show this using the long-division form?

$$\begin{array}{r|l} 5\overline{)50} & \\ & = \boxed{} \times 5 \end{array}$$

What is the missing numeral?" 10

The preceding example extends the basic division combinations to multiples of ten. They divide tens just as they divide ones. Let the pupils decide where to put the quotient, 10. Have them "check" their division by multiplying the divisor, 5, by the quotient, 10. Then, write the numeral 50 under the dividend, 50, and verify that the division is correct. The completed example should look like this:

$$\begin{array}{r|l} 10 & \\ 5\overline{)50} & \\ 50 & = \boxed{10} \times 5 \end{array}$$

Next, introduce the pupils to division using a three-digit numeral as a dividend, for example:

$$\begin{array}{r|l} 5\overline{)150} & \\ & = \boxed{} \times 5 \\ \hline & \end{array}$$

Ask, "If there are three 5's in 15, how many 5's are in 150? 30 Is it true that 30 × 5 = 150? Yes What is the missing numeral in our example? 30 Where do we write the product of (30 × 5)? Under the dividend in the example What numeral do we write above the dividend?" 30. The completed model should look like this:

$$\begin{array}{r|l} 30 & \\ 5\overline{)150} & \\ 150 & = \boxed{30} \times 5 \\ \hline & \end{array}$$

It should be emphasized that the quotient is obtained by dividing and that multiplication is the second step. Relate the combinations so that pupils will see 30 × 5 = 150; 150 ÷ 5 = 30; and 150 = $\boxed{30}$ × 5.

Have the pupils do some of the following for practice at their desks. Write these on the chalkboard.

(a)

$$\begin{array}{r|l} 60 & \\ 3\overline{)180} & \\ 180 & = \boxed{60} \times 3 \\ \hline & \end{array}$$

$$\begin{array}{r|l} 60 & \\ 5\overline{)300} & \\ 300 & 60 \times \boxed{5} \\ \hline & \end{array}$$

(b)

$$\begin{array}{r|l} 30 & \\ 6\overline{)180} & \\ 180 & 30 \times \boxed{6} \\ \hline & \end{array}$$

$$\begin{array}{r|l} 40 & \\ 4\overline{)160} & \\ 160 & \boxed{40} \times 4 \\ \hline & \end{array}$$

* * *

Pupil Exercises. Provide each pupil with worksheet No. 21.

Discuss this worksheet with the pupils before any independent work is undertaken. Encourage pupils to use a variety of "check" procedures to verify their answers.

DO EACH EXAMPLE USING THE LONG-DIVISION FORM. WRITE THE QUOTIENT ABOVE THE DIVIDEND.

	(a)	(b)	(c)
1.	$\begin{array}{r} 7 \\ 7\,\overline{)\,49} \end{array}$ $49 = \boxed{7} \times 7$	$\begin{array}{r} 8 \\ 8\,\overline{)\,64} \end{array}$ $64 = \boxed{8} \times 8$	$\begin{array}{r} 9 \\ 9\,\overline{)\,81} \end{array}$ $81 = \boxed{9} \times 9$
2.	$\begin{array}{r} 40 \\ 9\,\overline{)\,360} \end{array}$ $360 = \boxed{40} \times 9$	$\begin{array}{r} 30 \\ 4\,\overline{)\,120} \end{array}$ $120 = \boxed{30} \times 4$	$\begin{array}{r} 60 \\ 6\,\overline{)\,360} \end{array}$ $360 = \boxed{60} \times 6$
3.	$\begin{array}{r} 90 \\ 5\,\overline{)\,450} \end{array}$ $450 = \boxed{90} \times 5$	$\begin{array}{r} 90 \\ 7\,\overline{)\,630} \end{array}$ $630 = \boxed{90} \times 7$	$\begin{array}{r} 70 \\ 9\,\overline{)\,630} \end{array}$ $630 = \boxed{70} \times 9$

WRITE THE MISSING NUMERAL IN EACH BLANK SPACE:

	(a)	(b)	(c)
4.	350 ÷ 7 = 50	640 ÷ 8 = 80	540 ÷ 9 = 60
5.	490 ÷ 7 = 70	720 ÷ 8 = 90	369 ÷ 9 = 41

STUDY EACH EXAMPLE CAREFULLY, THEN COMPLETE THE NUMBER SENTENCE.

6. 549 ÷ 9 = (540 ÷ 9) + (9 ÷ 9)

= 60 + 1 or 61

7. 488 ÷ 8 = (480 ÷ 8) + (8 ÷ 8)

= 60 + 1 or 61

8. 637 ÷ 7 = (630 ÷ 7) + (7 ÷ 7)

= 90 + 1 or 91

LET'S PRACTICE.

	(a)	(b)	(c)
9.	275 - 68 = 207	75 - 68 = 7	4000 - 3697 = 303
10.	145 + 355 = 500	2400 ÷ 8 = 300	7 × 80 = 560

8. Dividing Tens and Ones Separately (Long-division form)

Objective. To develop skill in determining the quotient by using a two-step process.

Background Information for the Teacher. Dividing tens and ones separately in long-division form should now be introduced. The pupils are familiar with dividing tens and ones separately in horizontal form. Now, the objective here is to show that they can follow the same principle by extending the division algorithm.

The quotient is written in two-parts, illustration (a); one quotient is written on one line and the other quotient is written on the line below. These two quotients may be referred to as partial quotients, also, The relationship of the two-parts of the quotient in horizontal or sentence form,[illustration (b)],should be brought out. Help pupils to see that they must add the two partial quotients just as they added in the horizontal form. Now the importance of writing the quotient above the example as the final step becomes clear. Writing the quotient above the example both parallels the conventional placement, which will be used eventually, and provides the opportunity to combine the parts of the quotient into a single number.

(a)

$$\begin{array}{r|l} 12 & \\ 3\overline{)36} & \\ \underline{30} & = 10 \times \boxed{3} \\ 6 & \\ \underline{6} & = 2 \times 3 \\ & \end{array}$$

(b)

$$\begin{aligned} 36 \div 3 &= \boxed{12} \\ (30 + 6) \div 3 &= (30 \div 3) + (6 \div 3) \\ &+ \underline{10} + \underline{2} \\ &= 12 \end{aligned}$$

TEACHING PROCEDURE

Exploration. On the chalkboard write the following example and show the process and solution:

$$\begin{aligned} 24 \div 2 &= \boxed{} \\ (20 + 4) \div 2 &= (20 \div 2) + (4 \div 2) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

Next, write the same example in the long-division form so that it will appear like illustration (a):

(a)

$$\begin{array}{r|l} 2\overline{)24} & \\ & = 10 \\ \hline 4 & \\ & = 2 \\ \hline & \end{array}$$

(b)

$$\begin{array}{r|l} 12 & \\ 2\overline{)24} & \\ \underline{20} & = 10 \times 2 \\ 4 & \\ \underline{4} & = 2 \times 2 \\ & \end{array}$$

Ask, "Are there at least ten 2's in 24? yes What is 10 × 2? 20 Where shall we write the numeral 20 in our example? under the dividend, 24. What does 20 represent?" we subtracted ten 2's from 24.

"If we state that 24-20 = 4, what does the 4 represent? The new dividend. How many 2's are there in 4? 2 What is 2 × 2? 4 If we add the partial quotients 10 and 2, we will get 12. What does the number 12 represent?" There are twelve 2's in 24.

As the pupils respond to the questions, record the correct responses so that the complete algorithm will be like that of illustration (b).

In using another example, write the following problem on the board:

"At Logan Elementary School 55 boys asked to be on the basketball teams. Since only 5 players can be on any one team, how many teams will there be at the school?"

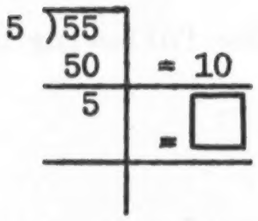
The teacher may wish to ask the following developmental questions: "Does the sentence 55 ÷ 5 = ☐ tell about the problem? yes Do you think there are enough boys for more than 10 teams? yes Why? because 10 X 5 = 50 Then, will the answer be a number larger than 10?" yes

Next, write the number sentence, 55÷5 = (50÷5) + (5÷5) on the chalkboard.

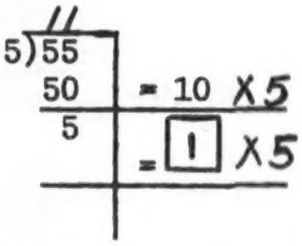
Ask, "How many steps are used to find the answer to the sentence on the board?" Two; (50÷5) = 10 and (5÷5) = 1.

Write the long-division form on the chalkboard, illustration (a):

(a)



(b)



Ask, "Are there at least ten 5's in 55? yes What is 10 × 5? 50 Why do we subtract, that is, 55-50? We already know one partial quotient. We want to determine the other partial quotient. How many 5's are there in 5? 1 What is 1 × 5? 5 (As the pupils respond to the above question, the correct responses should be recorded. Thus, the completed algorithm should look like illustration (b) above.)

The next example requires the use of basic combinations or the extension of basic combinations to dividing tens. The quotients are larger than those illustrated in previous examples. Emphasize the idea with pupils that they are to think of a combination they already know instead of "guessing." For instance, in dividing 248 by 4, there's no reason to begin with a partial quotient of 20 or 30 when they know there are six 4's in 24 and sixty 4's in 240.

Write the following examples on the chalkboard: *

(a)

$$248 \div 4 = \square$$

$$\begin{aligned}(240 + 8) \div 4 &= (240 \div 4) + (8 \div 4)\\ &= \underline{\qquad} + \underline{\qquad}\\ &= \underline{\qquad}\end{aligned}$$

(b)

$$\begin{array}{r|l} \multicolumn{1}{r}{62} & \\ 4\overline{)248} & \\ 240 & = 60 \times 4 \\ \hline 8 & \\ 8 & = 2 \times 4 \\ \hline \end{array}$$

With respect to example (a), point out that two-steps are required to arrive at the quotient, separating the tens as ones. In example (b), show that the same separation takes place; for example, ask "If there are six 4's in 24, how many 4's are in 240? 60 What is 60 × 4? 240 If we subtract 240 from 248, what is the difference? 8 Where is the difference, 8, written? In the ones' column below 240. How many 4's are there in 8? 2 What is 2 × 4? 8 In this long-division form, what have we determined?" There are 60 fours and 2 fours in 248; thus 60+2=62.

To provide written practice for the pupils, write the following examples. Have the pupils copy each example and complete the work.

(a)

$$\begin{array}{r|l} \multicolumn{1}{r}{83} & \\ 2\overline{)166} & \\ 160 & = 80 \times 2 \\ \hline 6 & \\ 6 & = 3 \times 2 \\ \hline \end{array}$$

(b)

$$\begin{array}{r|l} \multicolumn{1}{r}{32} & \\ 4\overline{)128} & \\ 120 & = 30 \\ \hline 8 & \\ 8 & = 2 \times 4 \\ \hline \end{array}$$

$$\begin{array}{r|l} \multicolumn{1}{r}{72} & \\ 4\overline{)288} & \\ 280 & = 70 \times 4 \\ \hline 8 & \\ 8 & = 2 \times 4 \\ \hline \end{array}$$

$$\begin{array}{r|l} \multicolumn{1}{r}{51} & \\ 5\overline{)255} & \\ 250 & = 50 \times 5 \\ \hline 5 & \\ 5 & = 1 \times 5 \\ \hline \end{array}$$

$$\begin{array}{r|l} \multicolumn{1}{r}{232} & \\ 2\overline{)464} & \\ 460 & = 230 \times 2 \\ \hline 4 & \\ 4 & = 2 \times 2 \\ \hline \end{array}$$

$$\begin{array}{r|l} \multicolumn{1}{r}{72} & \\ 3\overline{)216} & \\ 210 & = 70 \times 3 \\ \hline 6 & \\ 6 & = 2 \times 3 \\ \hline \end{array}$$

---

* As pupils respond to each question with respect to each example, record the responses as shown.

To check division, the number of tens and the number of ones in the quotient are multiplied by the divisor. The product is compared with the dividend. When the pupils have found, for example, that there are 62 fours in 248, remind them that they should multiply 62, that is, 62 × 4 = 248. Have them compare the example by analyzing the following illustration:

Factor → 62
Factor → × 4
Product → 248

Quotient → 62
Divisor → 4)248
Dividend → 248

Pupil Exercises. Provide each pupil with worksheet No. 22.

Encourage pupils to do neat and careful work. Many pupils will wish to work these exercises independently. Encourage all to do so if you wish, but observe each pupil's progress carefully and give help when necessary.

USE THE LONG-DIVISION FORM TO FIND THE QUOTIENT. CHECK EACH EXAMPLE BY MULTIPLICATION. THE FIRST EXAMPLE IS DONE FOR YOU.

(a)

1.
```
        51
   5 ) 255 |                 51
       250 | = 50 × 5      × 5
         5 |                255
         5 | =  1 × 5
           |
```

(b)
```
        51                  51
   6 ) 306 |              × 6
       300 | = 50 × 6      306
         6 |
         6 | =  1 × 6
           |
```

2. (a)
```
        21                  21
   8 ) 168 |              × 8
       160 | = 20 × 8      168
         8 |
         8 | =  1 × 8
           |
```

(b)
```
        91                  91
   7 ) 637 |              × 7
       630 | = 90 × 7      637
         7 |
         7 | =  1 × 7
           |
```

WRITE THE NUMERAL FOR THE LARGEST NUMBER THAT WILL MAKE EACH SENTENCE TRUE.

	(a)	(b)	(c)
3.	6 × 9 < 58	6 × 7 < 47	9 × 9 < 88
4.	8 × 9 < 80	6 × 5 < 32	5 × 8 < 47
5.	50 × 5 < 255	70 × 7 < 500	70 × 9 < 635

WRITE THE CORRECT NUMERAL FOR EACH SENTENCE.

	(a)	(b)	(c)
6.	6 × 7 = 42	6 × 70 = $\boxed{420}$	420 ÷ 6 = 70
7.	54 ÷ 6 = 9	540 ÷ 6 = 90	6 × $\boxed{90}$ = 540
8.	(4 × 10) + (4 × 3) = $\boxed{52}$	4 × 13 = 52	(2 × 13) + (2 × 13) = 52

9. Division with Remainders

Objective. To continue the use of the repeated subtraction form in division for the purpose of analyzing the division operation.

Background Information for the Teacher. The division operation is defined as finding the missing factor when the product and one factor are given. In the examples $24 = 6 \times \square$ or $\square = 24 \div 6$, 4 is the missing factor and thus satisfies the definition of the division operation.

If the number sentence to be solved is $80 = 6 \times \square$ (using only the set of whole numbers), the operation of division cannot be performed because there is no whole number which will satisfy the number sentence. However, $80 = (6 \times \boxed{13}) + 2$ will satisfy the condition. Thus, in this number sentence the division operation has not been performed since from the two numbers 80 and 6 there resulted not one number but two numbers--13 which is the missing factor and 2 which is the remainder.

Finding the missing factor $\square$ and the remainder $\triangle$ for sentences similar to $80 = (6 \times \square) + \triangle$ is the division process. For example, given any two whole numbers, a and b with b not equal to 0, the mathematical sentence describing the division process is:

$$a = (b \times \square) + \triangle$$

Example: $35 = (3 \times 11) + 2$

When the remainder is zero, the mathematical sentence describing the division process is:

$$a = (b \times \square) + 0 \qquad \text{or} \qquad \square = a \div b$$

Examples: $42 = (7 \times 6) + 0$ or $6 = 42 \div 7$

TEACHING PROCEDURE

Exploration. On the chalkboard represent the following illustration:

xxxxxxxxx
xxxxxxxxx
xxxxxxxxx
xxxx

As an introduction to the division algorithm with remainders, ask the following questions:

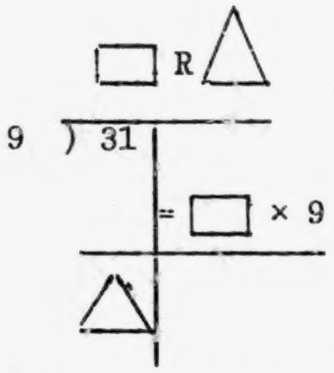
1. How many sets of 9 objects are represented on the chalkboard? 3
   Is there a set which does not have 9 members in it? Yes, one set has only 4 members.

2. Who could write a number sentence for the illustration on the chalkboard?
   $(3 \times 9) + 4 = \square$; $(3 \times 9) + 4 = 31$

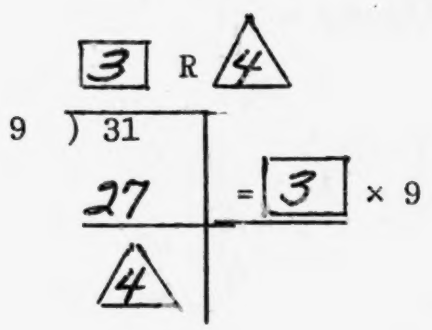
3. If there are 31 objects and 9 members in each set, what process may we use to find the number of complete sets of 9 objects? Division process How many complete sets of 9 objects are there? 3 How many objects remain? 4

Next, represent the illustration by using the algorithm in example (a) below. As the pupils respond to the following questions, record their responses so that the complete algorithm will look like example (b). Refer to the frames, □ and △ , as the "square-shaped frame" and the "triangle-shaped frame."

(a)



(b)

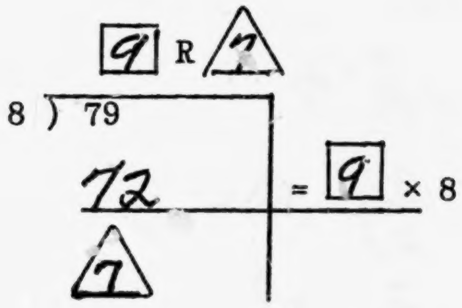


1. Is it true that 3 × 9 < 31? Yes Could there be 4 nines in 31? No If there are as many as 3 nines in 31, where should we put the 3? In or near the square-shaped frame.

2. What is 3 × 9? 27 Where shall we put the product of 3 × 9? Under the numeral 31 What is 31 - 27? 4 Where shall we put the numeral 4? In or near the triangle-shaped frame. What does the 4 represent? The remainder. How many members are in the incomplete set.

3. Where do we put the quotient 3 and the remainder 4 in our division form? Above the division form; the 3 replaces the square-shaped frame and the 4 replaces the triangle-shaped frame.

Next, use the following example with suggested questions to insure understanding and the relationship of the inequality sentences to division with a remainder.



1. Which of the following sentences should we use to find the quotient? Example (c)

(a) $7 \times 8 < 79$

(b) $8 \times 8 < 79$

(c) $9 \times 8 < 79$

What numeral will replace the square-shaped frame? 9
What is the product of 9 × 8? 72 Where do we write the numeral 72? Under the numeral 79 What is the replacement for the triangle-shaped frame? 7 Where do we write the quotient 9 and the remainder 7? Above the division form

2. Does the sentence $(9 \times 8) + 7 = 79$? Yes Explain. 9 is the quotient, 8 is the divisor, and 7 is the remainder; 79 is the dividend.

For individual practice (at the chalkboard and at their desks) have the pupils do the following, using the division process as illustrated in the foregoing examples:

(a)

$$\begin{array}{r|l} 7R3 & \\ 7\,\overline{)\,52} & \\ 49 & =7\times7 \\ \hline 3 & \end{array}$$

(b)

$$\begin{array}{r|l} 6R1 & \\ 8\,\overline{)\,49} & \\ 48 & =6\times8 \\ \hline 1 & \end{array}$$

(c)

$$\begin{array}{r|l} 4R6 & \\ 9\,\overline{)\,42} & \\ 36 & =4\times9 \\ \hline 6 & \end{array}$$

(d)

$$\begin{array}{r|l} 6R5 & \\ 9\,\overline{)\,59} & \\ 54 & =6\times9 \\ \hline 5 & \end{array}$$

(e)

$$\begin{array}{r|l} 9R5 & \\ 8\,\overline{)\,77} & \\ 72 & =9\times8 \\ \hline 5 & \end{array}$$

(f)

$$\begin{array}{r|l} 8R6 & \\ 7\,\overline{)\,62} & \\ 56 & =8\times7 \\ \hline 6 & \end{array}$$

To develop an appreciation of why it is best to use the largest number possible as the quotient, use illustration (a) to show several steps and illustration (b) to show the least number of steps. Introduce these illustrations by way of the following problem:

> "There are 7 children who must share 58 pieces of taffy candy. How many pieces should each child get?"

(a)

$$\begin{array}{r|l} 8R2 & \\ 7\,\overline{)\,58} & \\ 35 & =5\times7 \\ \hline 23 & \\ 21 & =3\times7 \\ \hline 2 & \end{array}$$

(b)

$$\begin{array}{r|l} 8R2 & \\ 7\,\overline{)\,58} & \\ 56 & =8\times7 \\ \hline 2 & \end{array}$$

Ask the following questions:

1. Suppose we used the sentence $5 \times 7 < 58$ for illustration (a). Is this sentence true? Yes What is the remainder when we subtract 35 from 58? 23 Are there more 7's contained in 23? Yes Did we find

the largest number of 7's in 58? No Since there are more 7's to be found, would we have to erase the work and start over, or could we continue? Continue; divide the remainder by 7.

2. Which of the sentences should we use to find the largest number of 7's in 23? Example (c)

(a)	(b)	(c)
$1 \times 7 < 23$	$2 \times 7 < 23$	$3 \times 7 < 23$

What is the remainder when we subtract, $23 - 21 = \square$? 2
Are there more 7's in 2? No How many 7's are contained in 58?
8 Is it true that $(5 \times 7) + (3 \times 7) = 56 + 2$? Yes

3. Let's work the example $7\overline{)58}$, illustration (b). What is the largest number of 7's contained in 58? 8 What is the remainder? 2 Is the remainder less than 7? Yes What is the complete quotient? 8 R 2 Which of the two illustrations uses the largest number possible as the quotient? Illustration (b)

Pupil Exercises. Provide each pupil with worksheet No. 23.

Make certain that pupils understand the instructions for examples 4, 5, and 6. Similar examples may be illustrated on the chalkboard and worked cooperatively before pupils are assigned independent work.

WRITE THE NUMERAL IN EACH FRAME FOR THE LARGEST NUMBER THAT WILL MAKE THE SENTENCE TRUE.

	(a)	(b)
1.	$\boxed{1} \times 9 < 17$	$\boxed{9} \times 7 < 68$
2.	$\boxed{7} \times 8 < 62$	$\boxed{9} \times 4 < 39$
3.	$\boxed{8} \times 8 < 71$	$\boxed{6} \times 8 < 55$

DO EXAMPLES (a) IN TWO STEPS AND EXAMPLES (b) IN ONE STEP (Using the largest number possible as the quotient).

	(a)*	(b)
4.	8R3 7) 59 35 = 5×7 24 21 = 3×7 3	7R7 8) 63 56 = 7×8 7
5.	4R4 9) 40 18 = 2×9 22 18 = 2×9 4	8R3 6) 51 48 = 8×6 3
6.	6R7 8) 55 40 = 5×8 15 8 = 1×8 7	9R2 9) 83 81 = 9×9 2

* Partial quotients will vary

10. Extending Division

Objectives. (1) To introduce quotients greater than 10. (2) To discover a shorter way of finding two-place and three-place quotients. (3) To use the distributive principle as an aid in determining partial products.

Background Information for the Teacher. The division algorithm for the division process where there is a quotient and a remainder is similar to that where there is a quotient with 0 as a remainder. Each requires the renaming of the product into addends which may be divided conveniently. The application of this form ensures that all numbers used are properly identified. However, the efficient use of the algorithm depends both on (a) knowing multiples so that they may be used in renaming the product and (b) renaming the product as the sum of those multiples so that a largest number (factor) is obtained for each place-value in one step. To illustrate, study the following example:

$$
\begin{array}{r|r}
144 \text{ R } 11 & \\
12\,)\overline{\,1739} & \\
1200 & = 100 \times 12 \\ \hline
539 & \\
480 & = 40 \times 12 \\ \hline
59 & \\
48 & = 4 \times 12 \\ \hline
11 & 144 \times 12
\end{array}
$$

Note that the multiples of 12 are 100 × 12 or 1200; 40 × 12 or 480; 4 × 12 or 48. The remainder is 11. In other words, the product of 1739 is renamed (1200 + 480 + 48) + 11 which makes division more meaningful. Using this distributive principle, each addend of 1739 is divided by 12, and the partial products are added to get the largest number or the quotient. The remainder is joined to the quotient to complete the process. For example, 1739 ÷ 12 = 144 R 11 may be illustrated in the following way:

Illustration (a): $1739 \div 12 = \boxed{144R11}$

$$
\begin{aligned}
(1200 + 480 + 48 + 11) \div 12 &= (1200 \div 12) + (480 \div 12) + (48 \div 12) + (11 \div 12) \\
&= \underline{100} + \underline{40} + \underline{4} + \underline{0} \text{ R } 11 \\
&= (100 + 40 + 4 + 0) + \text{R } 11 \\
&= 144 + \text{R } 11
\end{aligned}
$$

Illustration (b):

$$
\begin{array}{r}
\boxed{144R11} \\
12\,)\overline{\,1739\quad}
\end{array}
$$

$$
\begin{array}{r}
100 + 40 + 4 + 0 \text{ R } 11 \\
12\,)\overline{1200 + 480 + 48 + 11}
\end{array} =
$$

$$(100 + 40 + 4 + 0) + \text{R } 11 = 144 \text{ R } 11$$

TEACHING PROCEDURE

Exploration. Point out to the pupils that before a more detailed study of two-digit and three-digit quotients is attempted, it is important that the procedure is understood, where each numeral comes from, and where it should appear in the example. Moreover, develop an early appreciation for neat work in division, particularly in aligning the partial quotients.

Have the pupils copy the following example from the chalkboard and, step-by-step, guide their thinking by asking the following questions:

$$\begin{array}{r|l} \multicolumn{1}{r}{14} & \\ 6\,\overline{)\,84} & \\ 60 & = 10 \times 6 \\ \hline 24 & \\ 24 & = 4 \times 6 \\ \hline \triangle 0 & \end{array}$$

1. In this example, will the quotient be at least 10? Yes Can you think of the 84 as 60 + 24? Yes If we use 10 as a partial quotient, what will be the difference in 84 - 60? 24 When we divide again, what will be the next partial quotient? 4 What is 24 - 24? 0 When the remainder is 0, do we need to write it as a remainder? No

2. We found that there are 10 sixes and 4 more sixes in 84. What is the quotient? 14

3. Use this sentence to check the work:

$$14 \times 6 = \boxed{84}$$

$$\begin{aligned} (10 + 4) \times 6 &= (\underline{10} \times 6) + (\underline{4} \times 6) \\ &= \underline{60} + \underline{24} \\ &= \underline{84} \end{aligned}$$

Next, introduce the pupils to the following example which involves a remainder:

$$\begin{array}{r|l} \multicolumn{1}{r}{12\text{R}1} & \\ 8\,\overline{)\,97} & \\ 80 & = 10 \times 8 \\ \hline 17 & \\ 16 & = \boxed{2} \times 8 \\ \hline \triangle 1 & \end{array}$$

Ask the following questions:

1. Are there at least 10 eights in 97? Yes What is 10 × 8? 80 Where do we write 80? under 97 From what number will you subtract it? 97 What is the difference in 97 - 80? 17

2. We have found that there are 10 eights in 97. Are there still more eights in the difference which is 17? yes What will you put where the ☐ is? 2

3. What is the remainder? 1 What is the complete quotient? 12 R 1

For pupil practice, have them copy and complete each of the following examples. Point out that each partial quotient begins with 10.

(a)

14
7) 98
70 = 10 × 7
28
28 = 4 × 7
0

(b)

11R5
8) 93
80 = 10 × 8
13
8 = 1 × 8
5

(c)

12
9) 108
90 = 10 × 9
18
18 = 2 × 9
0

Since these examples are developmental, take time to insure understanding of the algorithm and the "checking" process. The "check" for the above examples is shown below:

(a) (14 × 7) + 0 = 98

(b) (11 × 8) + 5 = 93

(c) (12 × 9) + 0 = 108

Working with quotients greater than one 10 requires the use of the expression, for example, "tens of eightsl" This simply refers to some multiple of 10 times 8. The example "2 tens of eights" would mean 20 × 8; "3 tens of eights" would mean 30 × 8; etc. Also, try to lead pupils to discover the value of using a shorter method of division. Note illustrations (a) and (b) below:

(a)

25R4
8) 204
80 = 10 × 8
124
80 = 10 × 8
44
40 = $\boxed{5}$ × 8
4

(b)

25R4
8) 204
160 = $\boxed{20}$ × 8
44
40 = $\boxed{5}$ × 8
4

Write illustrations (a) and (b) on the chalkboard and ask the following questions:

1. In illustration (a) only 10 eights were used in the first step. Since 204 - 80 = 124, isn't 124 greater than 80? Yes Are there at least "1 tens of eight" more in 204? Yes Are there at least "2 tens of eight" in 204? Yes Complete the example and "check" by using this sentence: $(25 \times 8) + \triangle{4} = \square$. Is the product the same as the dividend in our illustration? yes How many steps were used to solve the example? Three

2. To solve the example in illustration (b) above, which of the following sentences will we use to reduce the number of steps? Example (b)

(a)	(b)	(c)
$10 \times 8 < 204$	$20 \times 8 < 204$	$30 \times 8 < 204$

Why would it be wrong to use example (c)? It is a false statement. How many steps will be needed to complete the example in illustration (b)? Two What is the remainder in each illustration? 4

For pupil practice, have them copy and complete the following examples. Have them begin the division by deciding if a quotient is at least 10; at least 20; or as much as 30.

(a)

$$\begin{array}{r|l} 28 & \\ 3\,)\overline{84} & \\ \underline{60} & = 20 \times 3 \\ 24 & \\ \underline{24} & = 8 \times 3 \\ 0 & \end{array}$$

(b)

$$\begin{array}{r|l} 24 & \\ 9\,)\overline{216} & \\ \underline{180} & = 20 \times 9 \\ 36 & \\ \underline{36} & = 4 \times 9 \\ 0 & \end{array}$$

(c)

$$\begin{array}{r|l} 31R2 & \\ 8\,)\overline{250} & \\ \underline{240} & = 30 \times 8 \\ 10 & \\ \underline{8} & = 1 \times 8 \\ 2 & \end{array}$$

(d)

$$\begin{array}{r|l} 25 & \\ 9\,)\overline{225} & \\ \underline{180} & = 20 \times 9 \\ 45 & \\ \underline{45} & = 5 \times 9 \\ 0 & \end{array}$$

(e)

$$\begin{array}{r|l} 28R1 & \\ 6\,)\overline{169} & \\ \underline{120} & = 20 \times 6 \\ 49 & \\ \underline{48} & = 8 \times 6 \\ 1 & \end{array}$$

(f)

$$\begin{array}{r|l} 30R3 & \\ 5\,)\overline{153} & \\ \underline{150} & = 30 \times 5 \\ 3 & \end{array}$$

Pupil Exercises. Provide each pupil with worksheet No. 24.

Make sure pupils understand the instructions before assigning independent work. With some less able pupils, work some of the exercises cooperatively to make certain that all concepts are carefully developed.

Interim Mathematics Unit - Grade 4
San Diego City Schools (1-65)

Worksheet No. 24

WRITE THE MISSING NUMERAL IN THE FRAME FOR THE LARGEST NUMBER OF TENS THAT WILL MAKE THE SENTENCE TRUE.

	(a)	(b)	(c)
1.	$\boxed{30} \times 8 < 243$	$\boxed{10} \times 9 < 168$	$\boxed{10} \times 8 < 152$
2.	$\boxed{20} \times 6 < 169$	$\boxed{20} \times 6 < 122$	$\boxed{20} \times 6 < 139$
3.	$\boxed{30} \times 9 < 273$	$\boxed{10} \times 8 < 159$	$\boxed{20} \times 8 < 162$

DIVIDE EACH OF THE FOLLOWING: (Each has a remainder.)

4. 6) 170 → 28 R 2
120 = 20×6
50
48 = 8×6
2

7) 185 → 26 R 3
140 = 20×7
45
42 = 6×7
3

9) 286 → 31 R 7
270 = 30×9
16
9 = 1×9
7

FIND THE QUOTIENT FOR EACH EXAMPLE IN ONE STEP:

5. 3) 120 → 40
120 = 40×3
0

7) 210 → 30
210 = 30×7
0

9) 360 → 40
3[illegible]0 = 40×9
0

6. 7) 350 → 50
350 = 50×7
0

9) 540 → 60
540 = 60×9
0

8) 560 → 70
560 = 70×8
0

GLOSSARY

ADDEND. The name given to any of the numbers to be added.

ADDITION. The process of joining two or more numbers and determining the total quantity. It is one of the four fundamental operations (processes) of arithmetic. Through addition we can find a sum without counting.

ALGORITHM. (Algorism). Any of many short-cut computational schemes or procedures in arithmetic, especially of those which require various, different or repeated steps. More generally, this term is applied to the form in which arithmetical operations are expressed. Thus $2\overline{)6}$ denotes the division algorism. The properties of arithmetic can be used to justify the procedures used in an algorithm.

ARRAY. An orderly arrangement of objects in rows and columns.

ASSOCIATIVE PRINCIPLE OF ADDITION AND MULTIPLICATION. Fundamental structural properties of elementary arithmetic referring to the order in which the operations of addition and multiplication are to be carried out when more than two numbers are involved. Examples are: $(3+4)+2 = 3+(4+2)$ and $(6 \times 2) \times 3 = 6 \times (2 \times 3)$. To generalize: $(a+b)+c = a+(b+c)$ and $(a \times b) \times c = a \times (b \times c)$.

BASE TEN. A system of numeration or a place-value system using ten as its fundamental counting group or base.

BINARY OPERATION. An operation which assigns to a pair of numbers (an ordered pair) another number which is called the sum in addition; the product in multiplication, etc.

CHECK. To verify the correctness of an answer or solution. It is not to be confused with "proof." When one checks the correctness of a sum or product by performing the operation in reverse order, one does not prove the result but merely seeks verification or added evidence for correctness.

COLUMN. A vertical arrangement of things, numerals, or any other symbols--most commonly used in "column addition." It is also the term which applies to the vertical arrangements in operations with arrays and tables, such as addition table, subtraction table, etc.

COLUMN ADDITION. A vertical arrangement of numerals for the purpose of facilitating the addition of more than two numbers. It is essential that the beginning learner become well acquainted with both the horizontal and vertical operational arrangements.

COMMUTATIVE PRINCIPLES OF ADDITION AND MULTIPLICATION. These are fundamental structural properties of elementary arithmetic which refer to the order in which the operations of addition and multiplication can or cannot be performed when only two numbers are involved. Examples are: $4+6 = 6+4$ and $3 \times 5 = 5 \times 3$. To generalize: $a+b = b+a$ and $a \times b = b \times a$.

CONCRETE. Tangible objects used to make an arithmetic situation more meaningful; the antithesis of abstract.

DIGIT. The basic numerals used in place-value systems of numeration. In the decimal system (base ten) the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The numeral 251 has three digits.

DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OVER ADDITION. One of the fundamental structural properties of arithmetic. An example is: $3 \times (4 + 3) = (3 \times 4) + (3 \times 3) = 12 + 9 = 21$. To generalize: $a \times (b + c) = (a \times b) + (a \times c)$, where a, b, and c are any numbers. This principle differs from the commutative and associative principles of multiplication in that it involves two of the four basic operations, that is, multiplication and addition.

DIVIDE. To divide one number by another means to find out how many times the second number can be subtracted from the first. For example, in $12 \div 4$, the procedure is to subtract 4 from 12 as many times as possible and then state as the quotient the number of times that the subtraction could be performed. It is the inverse of multiplication.

DIVISOR. The number in the division process by which another number is to be divided. In $15 \div 3 = 5$, the 3 is the divisor.

EQUAL VALUE. Two or more arithmetical expressions having the same numerical value though they may have very different forms.

EQUALITY. The general property indicated by two or more statements indicating equal value.

EQUIVALENCE. A relation that holds between two sets that can be put into one-to-one correspondence. For example: the set of days in one week is equivalent to the set of bright stars in the Big Dipper. Each set has seven members.

EXPANDED NOTATION. A numeral that shows the number represented by each digit in a place-value numeral. For example, the expanded notation for the numeral 379 is: $300 + 70 + 9$, or $(3 \times 100) + (7 \times 10) + (9 \times 1)$.

FACTOR. Any of the numbers to be multiplied to form a product. Thus, in $3 \times 4 = 12$, the 3 and 4 are factors of 12. For beginning learners it is wise to insist on "factors" or "factor" rather than on "multiplier" and "multiplicand."

IDENTITY ELEMENT. A number and its symbol which does not change the value when a certain operation is performed. Zero is called the identity element of addition; "one" is called the identity element of multiplication.

IDENTITY. An equation like $5 = 5$ in which the values on either side of the = sign are identically the same. A principle such as the commutative principle of addition, $a + b = b + a$, is an identity since it is true for all numbers a and b.

INTERSECT. To meet or cross in a point or confined space.

INVERSE (OF AN OPERATION). The term should be restricted in its elementary use to operations. Subtraction is the inverse of addition; division is the inverse of multiplication. For beginning learners, the idea of operational inverse may be conveyed by showing that one operation does what its inverse undoes.

LINE SEGMENT. A unit of definite length which can be described in appropriate units of measure.

MULTIPLE. A term indicating a product obtained by the multiplication of a specified number and any whole number. If the specified number is 4, for example, we speak of the series 4, 8, 12, 16, etc., and the "multiples of 4." The term "multiple" should replace, where possible, the term "multiplication facts."

MULTIPLICAND. Thinking of multiplication as repeated additions, that factor which is to be added a specified number of times is the multiplicand. Thus, in 4 × 5, the factor 5 is the multiplicand.

NUMERALS. Symbols used to denote numbers, as the Hindu-Arabic or the Roman numerals. Numerals are concrete; they can be written on the chalkboard. The numbers which they denote or for which they stand are abstract; they can only be represented by symbols.

OPERATION. In elementary arithmetic this refers to an "action" relation; the process of carrying-out the rules of a procedure or of an algorism. The four fundamental operations are: addition, subtraction, multiplication, and division.

PARTIAL PRODUCT. Literally, a part of a product but not the "whole product." It is used in elementary arithmetic with regard to the written algorithm of multiplication. In this algorithm each digit in the multiplier produces one partial product. The final product is then the sum of the partial products.

PLACEHOLDER. Any symbol (frame, letter, etc.) designed to hold a place in a number or mathematical expression. In the contemporary approach to elementary arithmetic, "placeholder equations" play an important role, especially with younger pupils.

PRINCIPLE. A property of numbers is a fact or characteristic that is always true for a specified situation involving numbers. (See commutative principles of addition and multiplication, associative principles of addition and multiplication, etc.). Synonym for property.

RENAMING. The process of replacing a numeral by a different numeral to name the same number. For example, the numeral 256 names a particular number; 250 + 6; 200 + 50 + 6; 200 + 40 + 16; and 16 × 16 also name 256.

REPEATED ADDITION: The "long form" of multiplication.

RESULT. The final outcome of an operation or algorithm. Whenever possible, one should insist on calling a result by its specific name, that is, "sum," "product," "missing addend," "quotient," etc.

ROW. A line or arrangement running horizontally. For example, the entries in a multiplication table which run from left-to-right are called "rows."